DYNAMICS

(FORMERLY KNOWN AS 'INTERMEDIATE DYNAMICS')

Revised for Three-Year Degree Course

(Pass and Honours)

&

Intermediate Course

BY

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PREFACE

THIS book, as its name indicates, is meant to be a text-book for the Intermediate students, both Arts and Science, of the Indian Universities and various Education Boards. Regarding the subject-matter, we have tried to make the exposition clear and concise without going into unnecessary details. Varied types of examples have been worked out by way of illustrations in each chapter and the examples set for exercise have been carefully selected and properly graded.

Important formulæ and results have been given at the beginning of the book for ready reference. Questions of the University of Calcutta and some other Universities are given at the end, to give the students an idea of the standard of the examination.

Our best thanks are due to Prof. K. N. Chakravarty and Prof. G. D. Bhar of Presidency College, Prof. N. L. Ghosh of Ripon College, Prof. P. Roy of Berhampur K. N. College, Prof. G. D. Mukherjee of Ashutosh College, Prof. D. K. Dey of Mohosin College, Hooghly, Prof. N. Karforma of Scottish Church College and others for their valuable criticisms and helpful suggestions.

Any criticism, correction and suggestion towards improvement from teachers and students will be thankfully received.

CALCUTTA
June, 1946

B. C. D.
B. N. M.

Syllabus for

ELEMENTARY DYNAMICS

Three-Year Degree Course

Pass [30 marks]

KINEMATICS : Position, displacement, speed, velocity and acceleration of a particle. Laws of composition and resolution : Parallelogram, Triangle and Polygon of velocities. Relative velocity. Motion in a straight line under constant acceleration.

KINETICS : Newton's Laws of motion, measure of a force, gravitational units. Vertical motion under gravity (in vacuum), particle moving down smooth and rough inclined planes and motion of a connected system.

IMPULSE, WORK AND ENERGY : Definitions and basic theorems. Conservation of energy and linear momentum. Impulsive forces. Power of engines and pumps. Projectiles, range on horizontal and inclined planes.

PRACTICAL PROBLEMS : Walking, friction on the wheels of an engine and a train.

Simple Harmonic motion. Forced oscillation with linear damping. Uniform motion in a circle. (Use of Calculus advised.)

Honours [Statics and Dynamics 50 marks]

KINEMATICS : Position, displacement, velocity, speed, acceleration. Composition and resolution of velocities and acceleration. Relative velocity. Normal acceleration.

KINETICS : Newton's Laws of motion. Measurement of force. Different systems of units. Simple applications ; motion over pulleys, on inclined planes (smooth and rough). Problems of walking, friction on moving wheels. Vertical motion under gravity, parabolic motion under gravity. Range and maximum range of a projectile under gravity. Impulse, work, energy ; Definitions and basic theorems. Principles of conservation of linear momentum and energy. Impulsive forces. Power of engines and pumps. Collision of elastic bodies. Newton's laws, impact of a sphere on a plane, impact of two spheres. Energy loss.

Syllabus for
DYNAMICS
of Gauhati & Dibrugarh Universities

Pass :

Velocity and acceleration ; laws of composition and resolution of velocities and accelererations ; relative velocity ; force ; simple harmonic motion ; motion in two dimensions ; projectiles ; impulse, work, energy ; impulsive forces ; conservation of energy and linear momentum ; power of engines and pumps ; impact of elastic bodies.

(Use of *Calculus* advised)

Honours :

Particle moving down smooth and rough inclined planes and motion of a connected system ; tangential and normal accelerations ; impact of elastic bodies ; uniform motion in a circle.

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Important Formulæ and Results

1. Resultant of two Velocities.

$$w = \sqrt{u^2 + v^2 + 2uv \cos \alpha}$$

$$\theta = \tan^{-1} \frac{v \sin \alpha}{u + v \cos \alpha}.$$

2. Rectilinear Motion with Uniform Acceleration.

$$v = u + ft.$$

$$s = ut + \tfrac{1}{2}ft^2.$$

$$v^2 = u^2 + 2fs.$$

Space described in the t^{th} sec.

$$s_t = u + \tfrac{1}{2}f(2t - 1).$$

Average velocity $= \frac{1}{2}(u + v)$

$= u + \frac{1}{2}ft.$

3. Rectilinear Motion.

$$\text{Veloc. } v = \frac{dx}{dt} : \text{acceln. } f = \frac{dv}{dt} = \frac{d^2x}{dt^2} = v\frac{dv}{dx}.$$

4. Vertical Motion under Gravity.

$g = 32$ ft. per sec^2, or, 981 cms. per sec^2.

For upward motion

$$\text{Greatest height} = \frac{u^2}{2g}$$

$$\text{Time to the greatest height} = \frac{u}{g}$$

$$\text{Total time of flight} = \frac{2u}{g}.$$

For downward motion

$$v = \sqrt{2gh}, \quad t = \sqrt{\frac{2h}{g}}.$$

5. Motion on a plane.

$$v_x = \frac{dx}{dt}, \quad v_y = \frac{dy}{dt}, \quad f_x = \frac{d^2x}{dt^2} = \frac{dv_x}{dt}, \quad f_y = \frac{d^2y}{dt^2} = \frac{dv_y}{dt}.$$

6. Laws of Motion.

$$P = mf$$

$$W = mg$$

$$P = m\frac{d^2x}{dt^2} = m\frac{dv}{dt} = mv\frac{dv}{dx}$$

$$X = m\frac{d^2x}{dt^2}, \quad Y = m\frac{d^2y}{dt^2}.$$

7. Projectiles.

Greatest height $= \dfrac{u^2 \sin^2 a}{2g}$

Time to the greatest height $= \dfrac{u \sin a}{g}$

Total time of flight $= \dfrac{2u \sin a}{g}$

Horizontal Range $= \dfrac{u^2 \sin 2a}{g}$

Maximum Horizontal Range $= \dfrac{u^2}{g}$ when $a = 45°$

Latus rectum of the path $= \dfrac{2u^2 \cos^2 a}{g}$.

Velocity at height h, $v^2 = u^2 - 2gh$

Height of the directrix $= \dfrac{u^2}{2g}$.

Distance of the focus from the point of projection
= height of the directrix.

Range *up* an inclined plane (inclination β)

$$R = \frac{u^2}{g\cos^2\beta}[\sin(2a-\beta) - \sin\beta]$$

Max. oblique range *up* $= \dfrac{u^2}{g(1+\sin\beta)}$

Range *down* an inclined plane (inclination β)

$$R = \frac{u^2}{g\cos^2\beta}\,[\,\sin(2\alpha-\beta)+\sin\beta\,]$$

Max. oblique range *down* $= \dfrac{u^2}{g(1-\sin\beta)}$.

Analytical equation of the path of a projectile

$$y = x\tan\alpha - \tfrac{1}{2}g\frac{x^2}{u^2\cos^2\alpha}$$

Co-ordinates of the vertex $A\left(\dfrac{u^2\sin 2\alpha}{2g}, \dfrac{u^2\sin^2\alpha}{2g}\right)$

Co-ordinates of the focus $S\left(\dfrac{u^2\sin 2\alpha}{2g}, -\dfrac{u^2\cos 2\alpha}{2g}\right)$

Polar co-ordinates of the focus $S\left(\dfrac{u^2}{2g}, 2\alpha-\dfrac{\pi}{2}\right)$

8. Work, Power and Energy.

Horse-Power = 550 foot-pounds per sec.

Kinetic Energy $= \frac{1}{2}mv^2$

Potential Energy $= mgh$ (at height h).

9. Impulsive Forces.

Impulse of a force $= Pt$

$$= m(v-u),$$

$$\int_{t_0}^{t} P.dt = \int_{u}^{v} m\frac{dv}{dt}\,dt = mv - mu.$$

10. Collision of Elastic Bodies.

Principle of conservation of linear momentum

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$$

along the line impact

Newton's Law for direct impact

$$v_1 - v_2 = -e(u_1 - u_2).$$

Loss of K.E. by direct impact

$$=\frac{1}{2}\frac{m_1m_2}{m_1+m_2}(1-e^2)(u_1-u_2)^2.$$

For equal loss of K.E. (by direct impact) of each sphere $u_1+u_2+v_1+v_2=0$.

11. Angular Velocity and Normal Acceleration.

$$\text{Angular velocity}=\omega=\frac{d\theta}{dt}=\frac{pv}{r^2}$$

$$\text{for a circle } \omega=\frac{v}{r}.$$

$$\text{Normal acceleration}=\frac{v^2}{r}=\omega^2r.$$

12. Simple Hamonic Motion.

$$\text{Acceleration}=\frac{d^2x}{dt^2}=-\mu x$$ (x = distance from the centre of oscillation $\mu > 0$)

$$\text{Velocity}=\frac{dy}{dt}=\pm\sqrt{\mu(a^2-x^2)}$$

(a being the amplitude)

$$x=a\cos(\sqrt{\mu}t+\varepsilon).$$

$$\text{Period}=T=2\pi/\sqrt{\mu}.$$

13. Simple Pendulum.

$$\text{Time of oscillation}=T=2\pi\sqrt{\frac{l}{g}}.$$

14. Hooke's Law.

$$T=\lambda\frac{x}{l}.$$

DYNAMICS

CHAPTER I

INTRODUCTION

1·1. Scope and divisions of the subject.

Mechanics is the science concerning the conditions of rest or motion of the objects around us.

That branch of the subject which deals with bodies at rest when acted on by forces, or more properly, with the relations between the forces which acting on a body keep it at rest, is called **Statics.**

Dynamics is that branch of the subject which treats of the motions of bodies under the action of forces.

This is again subdivided into two parts :

The first part called **Kinematics** or *Phoronomy* deals with the different types of motions possible for a body, and the effects thereof on its position, in fact, the geometry of the motions, without enquiring into the causes which produce these motions.

The second part investigates the laws of motions, that is, the relations existing between the forces acting on a body, and the motions produced thereby, and is known as **Kinetics** or *Dynamics* proper.

1·2. Motion.

Motion is change of position. When a body is changing its position,* we say that it is in *motion.* When it is not changing its position, it is said to be at rest.

---

*For detailed discussion on this point see Art. 3·1.

Now a material body like a book or a stick, while changing its position, may move either as a whole from one place to another, or else may turn or rotate near about the same position. The general motion of a body is a combination of both these types of motion and the investigation of such motions is reserved for more advanced treatises on Dynamics. In this elementary work on the subject, as a first step, we shall confine ourselves to the consideration of motion of 'particles' or 'material points' only, for which motion would mean bodily transference from one point to another in space, and no question of rotation about an axis in itself or spin, as it is called, arises.

1·3. Definitions.

In the above articles we have used certain terms with which we are familiar from common use ; now we proceed to define them formally.

Matter is anything that occupies space and can be perceived by our senses.

A **body** is a portion of matter limited in all directions, having a finite shape and size, and occupying some definite space.

A **particle** is a portion of matter which is indefinitely small in size, so small, that for the purpose of our investigation, the distance between the different parts of it may be neglected. From a dynamical point of view it is considered as a *material point* for which rotation or spin has no meaning, and any motion of it signifies a transference from one point of space which it practically occupies, to another.

Force is something which changes or tends to change the state of rest or of uniform motion of a body.

[This definition will be more fully discussed in Chapter VI.]

Mass is the quantity of matter in a body. It is distinct from the bulk occupied by the body *i.e.*, its volume. For example, a piece of cotton can be squeezed to occupy a smaller

volume, but its mass, that is the quantity of matter in it, is the same as before.

F.P.S. and C.G.S. units.

The *units* for measuring *length*, *mass* and *time*, commonly used in England, are *foot*, *pound* and *second* respectively, and this system of units is generally referred to as the **F.P.S.** system.

In many other countries, as also in scientific measurements, the units used for length, mass and time are respectively *centimetre*, *gramme* and *second*, and this system is referred to as the C.G.S. system.

The relations between the two systems are :

1 foot = 30·4 centimetres (nearly)

1 inch = 2·54 cms.

1 lb. = 453·6 gms. (roughly)

CHAPTER II

SPEED AND VELOCITY

2·1. Definitions.

Speed—*The rate at which a moving point traces out its path is called its speed.*

Uniform Speed—*The speed of a moving point is defined to be uniform when it passes over equal lengths of its path in equal intervals of time, however small these equal time intervals may be taken.*

Note. Suppose that a particle describes 10 feet of its path in each second. Here the speed of the moving point may or may not be uniform, for it is quite possible that in each second, during the first half, the particle may move over 6 feet, and during the latter half, over only 4 feet, making up a total of 10 feet in each second considered. Even if it traces out 5 feet in each half second, its speed may not be uniform, for there may be variations within the half second, though the total may be the same in each case. Hence the utility of the last clause of the above definition.

When a particle is moving with uniform speed, its speed is measured by the ratio of the total length of its path traced out in any interval of time, to that time. It is the same always.

Non-uniform speed ; Speed at a point—Consider a moving particle whose speed is variable. For instance, an engine starting from a station moves slowly at first, and then quickens its speed, and again slows it down when approaching the next station. We may make an estimate of the comparative quickness or slowness of its motion at two particular moments, and at one instant we may find its motion quicker, or speed greater, than at another moment. It is quite conceivable that at a particular moment, when at a particular point of its path, it is moving with high speed,

but even after the lapse of a quarter of a second, its speed may come down considerably. Hence a necessity arises of defining *speed at any point* in such a case.

When the speed of a moving point is non-uniform, its speed at any instant during its motion, *i.e.*, at a particular point of its path, is measured by the ratio of the distance travelled in a very short period of time including the instant in question, to that time, provided the time interval is infinitely small (so small, that any change of speed during the interval may be neglected).

Let a point be moving along a path AB. While at any point P of its path, let it describe a small distance PP' $(=s')$ in a very small time t'. Then its speed at P is measured by $\frac{s'}{t'}$, ultimately or more correctly by the limiting value of the ratio $\frac{s'}{t'}$ when t' is infinitely small.

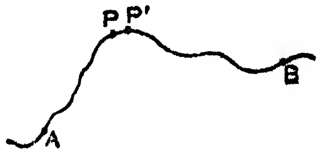


Fig. (i)

It is clear that the speed at P is not necessarily given by the distance actually travelled in a unit of time from P, for its speed may change, or it may even come to rest, before the unit time is elapsed. In fact, the speed at P measures the distance which the particle would pass over in a unit time, if it continued to move throughout that time at the same rate as at P.

Average Speed—The average speed of a moving point for any prescribed interval of time during its motion, is that uniform speed with which the point would describe the same length of its path as actually passed over, in the same interval of time.

Mathematically speaking, if a length AB $(=s)$ of its path is described in an interval t, the average speed during the interval is $\frac{s}{t}$.

Displacement—The displacement of a moving point in any time is its change of position, as indicated by the straight line joining its initial and final positions during the interval.

Displacement thus involves the idea of both magnitude and direction, *i.e.*, the distance through which the moving point is displaced, and the direction of the straight line in which it is on the whole displaced, no matter even if by a curved path. In figure (i), as the particle traces out its path from A to B, the displacement is the straight length AB, in that direction.

Velocity—*The velocity of a moving point is its rate of change of position, (or rate of displacement) having a definite direction and magnitude.*

Uniform velocity—*The velocity of a moving point is said to be uniform, when it always moves along the same straight line in the same sense, and passes over equal distances in equal intervals of time, however small these time intervals may be taken.*

Distinction between Uniform Velocity and Uniform Speed.

For uniform velocity, the moving point must always move along the same direction, which need not be the case for uniform speed, though in both cases, equal lengths of the path should be described in equal intervals of time, however small. For instance, a point moving uniformly in a circle (*e.g.*, the free extremity of the hand of a clock) has uniform speed, but not uniform velocity. It may be noted that a point moving with uniform velocity must have its speed also uniform, but a point moving with uniform speed need not have a uniform velocity, as illustrated in the example above.

Non-uniform velocity ;- velocity at a point—In case when the velocity of a moving point is non-uniform, the velocity at any point P of its path [See *Fig.* (*i*)] has got its magnitude determined by the limiting value of the ratio $\frac{s'}{t'}$, where s' is the displacement PP' in an infinitely small time t' including the instant when the particle is

at P, and *the direction of the velocity* there, being the ultimate direction of PP' as P' comes infinitely close to P, *is along the tangent at P to the path.**

Average velocity—The average velocity of a moving point during any interval of time is that uniform velocity with which a particle would receive the same displacement in the same time, as that of the given point.

Mathematically, if d is the length of the straight line joining the initial and final positions corresponding to the interval t, the average velocity is $\frac{d}{t}$, in the direction of that line.

It may be noted that in case of curved path the magnitude of the average velocity is different from the average speed during any interval.

2·2. Vector and Scalar quantities.

A quantity which has got both magnitude and direction (in a definite sense) is defined to be a *vectar* quantity ; *e.g.*, velocity, displacement, force, acceleration, etc.

A quantity which has a magnitude, but is not associated with any direction, is defined to be a *scalar* quantity ; *e.g.*, speed, mass, temperature, etc.

As a straight line has got a magnitude, direction and sense, *any vector quantity may very aptly be represented by a straight line*, the length of the line representing the magnitude of the vector on a suitably chosen scale, the direction of the straight line giving the direction of the vector, and the sense of the vector being indicated by an arrow-mark along the line.

Thus the velocity of a moving point or a force acting at a point on a body, may very well be represented in magnitude and direction by a straight line.

* For analytical expression of velocity see later.

For the sake of convenience, *a velocity represented in magnitude, direction and sense by AB is often denoted by* $\overrightarrow{AB}$ or $\mathbf{AB}$.

2·3. Simultaneous velocities of a point : Resultant velocity and components.

Let an insect be moving on a table on board a ship. If the ship be in motion on water, the table, along with the insect on it, shares the motion in common with the ship, and the insect has thus two simultaneous motions. If in addition the table be dragged on the ship in another direction, the insect also moves with the table, and it has thus three motions superposed on it. In this way we may conceive of a number of simultaneous motions of the same particle.

If a particle possesses several simultaneous velocities in different directions due to various reasons, and if the joint effect be the same as if the particle moves with a single velocity in a definite direction, this latter velocity is known as the *resultant* of the given simultaneous velocities which, in their turn, are called the *component* velocities of the single resultant.

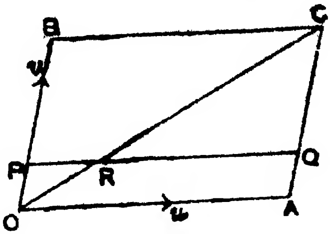
2·4. Parallelogram of velocities.

If a point has simultaneously two velocities which are represented in magnitude and direction by the two adjacent sides of a parallelogram meeting at an angular point, then the resultant velocity of the particle is represented in magnitude and direction by the diagonal of the parallelogram drawn from that angular point.

Let a point O possess simultaneously two velocities u and v, represented in magnitude and direction by the two adjacent sides OA and OB of the parallelogram $OACB$. Join OC.

The simultaneous existence of the two velocities of the particle may be imagined by supposing that the particle

moves along OA with a velocity represented by OA, and at the same time, the line OA, with the particle on it, moves, remaining parallel to itself, so that the end O moves along OB with a velocity represented by OB.



At the end of a unit of time, owing to the velocity u, the particle reaches the extremity A, while on account of the velocity v, the line OA takes up the parallel position BC, so that, at the end of a unit of time, the final position of the particle is at C.

At any intermediate instant, say after $1/n$th of a second from start, due to the velocity v, the line OA takes up the parallel position PQ, where $OP=(1/n).OB$. At the same time, due to the velocity u, the distance of the particle described from P along PQ is $(1/n).OA$.

Now, if R be the point of intersection of PQ and OC, since PR is parallel to BC,

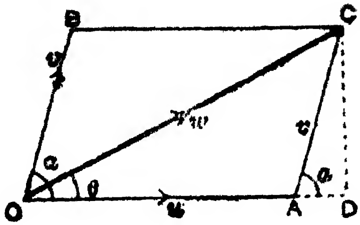
$$\frac{PR}{BC}=\frac{OP}{OB}=\frac{1}{n}, \text{ and so } PR=\frac{1}{n}\cdot BC=\frac{1}{n}\cdot OA.$$

Hence, from what is stated above, the position of the particle after $1/n$th second from start is at R on OC, where $\dfrac{OR}{OC}=\dfrac{OP}{OB}=\dfrac{1}{n}$.

As n may have any value, we find that due to the two simultaneous velocities, u and v, the position of the particle is all along on the line OC, and is such as if the particle moves with a single velocity represented by OC.

OC thus represents the resultant velocity in magnitude and direction.

25. Analytical expression for the resultant of two given velocities.



Let u and v be the two velocities of a particle O in directions OA and OB inclined at an angle α. Let them be represented by OA and OB. Complete the parallelogram $OACB$, and join the diagonal OC, which then, by parallelogram of velocities, represents the resultant velocity w. Let $\angle COA = \theta$ which will give the direction of the resultant. Now draw CD perpendicular upon OA (produced if necessary). Then from the right-angled triangle CAD, $AD = AC \cos CAD = v \cos \alpha$, and $CD = AC \sin CAD = v \sin \alpha$.

Thus, $OC^2 = OD^2 + CD^2$ gives

$$w^2 = (u + v \cos \alpha)^2 + (v \sin \alpha)^2$$
$$= u^2 + 2uv \cos \alpha + v^2.$$

Also $\tan \theta = \dfrac{CD}{OD} = \dfrac{v \sin \alpha}{u + v \cos \alpha}$.

Hence, $\mathbf{w} = \sqrt{\mathbf{u}^2 + 2\mathbf{uv} \cos \alpha + \mathbf{v}^2}$

and $\theta = \tan^{-1} \dfrac{\mathbf{v} \sin \alpha}{\mathbf{u} + \mathbf{v} \cos \alpha}$

give the magnitude and direction of the resultant.

Cor. 1. If $\alpha = 0$, $\mathbf{w} = \mathbf{u} + \mathbf{v}$ and if $\alpha = \pi$, $\mathbf{w} = \mathbf{u} - \mathbf{v}$.

Hence, *the resultant of two simultaneous velocities along the same line is their algebraic sum.*

Again, if $\alpha = 90°$, $\mathbf{w} = \sqrt{\mathbf{u}^2 + \mathbf{v}^2}$.

Cor. 2. When $v=u$, it is easily seen that

$$w=2u\cos\frac{\alpha}{2} \text{ and } \theta=\frac{\alpha}{2}.$$

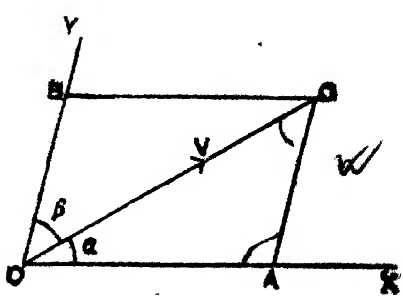
Thus, *the resultant of two equal velocities u, u, at an angle α, is $2u\cos\frac{\alpha}{2}$ in a direction bisecting the angle between them.*

2·6. Breaking up a given velocity into two components.

A given velocity may be resolved into two components in an infinite number of ways, for by parallelogram of velocities, if with the straight line representing the given velocity as diagonal we construct *any* parallelogram, the two adjacent sides of this parallelogram will represent the two component velocities having the given velocity as their resultant.

Again, if we want the component of a given velocity in a given direction at any inclination to it, the component is not determinable, in as much as the direction of the other component may be chosen to be anything, and the parallelogram constructed with the given velocity as diagonal.

If, however, with a given velocity, both the directions are definitely mentioned in which we are required to break it up into components, these components can be determined.



Let $OC\,(=V)$ represent the given velocity, and OX and OY, given directions making angles α and β to it on either side, in which we are to resolve it into components.

Complete the parallelogram $OACB$ with diagonal OC and sides along OX and OY. Then by parallelogram of velocities,

OA and OB represent the required components of V along OX and OY.

Now from triangle OAC, by Trigonometry,

$$\frac{OA}{\sin OCA} = \frac{AC}{\sin COA} = \frac{OC}{\sin OAC}$$

i.e., $$\frac{OA}{\sin \beta} = \frac{OB}{\sin \alpha} = \frac{V}{\sin (180° - \alpha + \beta)} = \frac{V}{\sin (\alpha + \beta)}.$$

$$\therefore \quad OA = \frac{V \sin \beta}{\sin (\alpha + \beta)}, \quad OB = \frac{V \sin \alpha}{\sin (\alpha + \beta)}.$$

2·7. Resolving a given velocity into two perpendicular components.

The most important case of resolution of a given velocity into two components is, when the directions of the components are at right angles to one another. In this case the components are referred to as *resolved parts* in

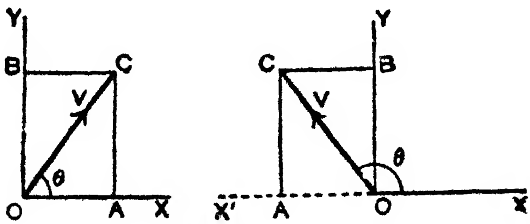


Fig. (i) Fig. (ii)

the corresponding directions. Let the given velocity V be represented by OC, and let the direction OX make an angle $COX = \theta$ with it, OY being perpendicular to OX.

Completing the parallelogram $OACB$ (which is a rectangle in this case), we notice that the required resolved parts along OX and OY are given by

$$OA = OC \cos XOC = \mathbf{V} \cos \theta,$$

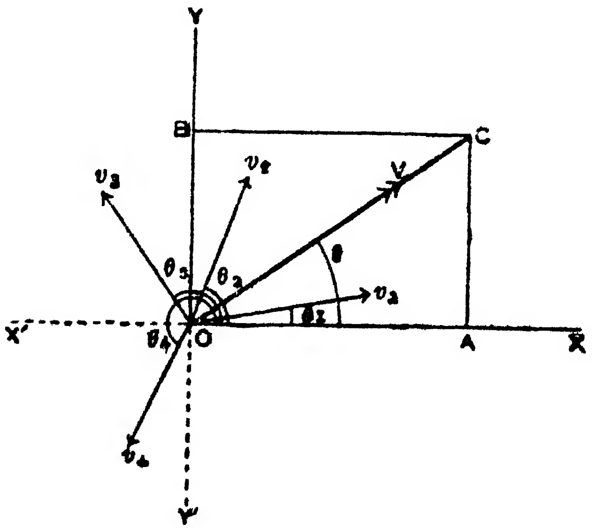
and $$OB = AC = OC \sin XOC = \mathbf{V} \sin \theta.$$

Note. In Figure (ii), OA, strictly speaking, is $OC \cos COA = V \cos (180° - \theta) = -V \cos \theta$, and is positive along OX'. Now, *mathematically a velocity u along OX' is identical with a velocity $-u$ along OX.* Hence, $-V \cos \theta$ along OX' may be described as $V \cos \theta$ along OX.

Thus, the resolved part of V along OX is mathematically $V \cos \theta$, whether θ be obtuse or acute or anything.

Hence, *any given velocity V is mathematically equivalent to (and accordingly can be replaced, whenever needed, by) two simultaneous resolved parts, one, $V \cos \theta$ along a direction OX at angle θ to it, and another, $V \sin \theta$ perpendicular to it,* whatever the angle θ may be. This mode of replacing a given velocity by its two equivalent resolved parts in two suitable perpendicular directions is particularly useful in finding the resultant of several simultaneous velocities for a particle, as is shown below.

2·8. Resultant of several simultaneous coplanar velocities of a particle.



Let a point O possess several simultaneous velocities v_1, v_2, v_3,... etc. in different given directions in the same plane

and let their directions make angles θ_1, θ_2, θ_3,... with any suitable chosen direction OX in the plane, OY being perpendicular to OX.

We can replace the velocity v_1 by its components $v_1 \cos\theta_1$ along OX, and $v_1 \sin\theta_1$ along OY. Similarly, v_2 may be replaced by $v_2 \cos\theta_2$ along OX, and $v_2 \sin\theta_2$ along OY, and so on for each one of the given velocities.

The given simultaneous velocities are then mathematically equivalent to a single total component $v_1 \cos\theta_1 + v_2 \cos\theta_2 + v_3 \cos\theta_3 + \cdots$ along OX, and a single total component $v_1 \sin\theta_1 + v_2 \sin\theta_2 + v_3 \sin\theta_3 + \cdots\cdots$ along OY.

These two final components along OX and OY, (represented by OA and OB say,) will be equivalent to the resultant velocity V represented by OC at an angle θ to OX, where

$$V \cos\theta = OA = v_1 \cos\theta_1 + v_2 \cos\theta_2 + v_3 \cos\theta_3 + \cdots$$

$$V \sin\theta = OB = v_1 \sin\theta_1 + v_2 \sin\theta_2 + v_3 \sin\theta_3 + \cdots$$

From these, V and θ are definitely determined.

Note. From the above, squaring and adding, we ultimately get

$$V = \sqrt{\Sigma v_1^2 + 2\Sigma v_1 v_2 \cos(\theta_2 \sim \theta_1)},$$

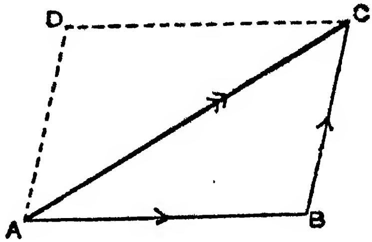
the positive sign of the square root being taken for the magnitude of V, and then the signs of the right-hand expressions will give the signs of $\sin\theta$ and $\cos\theta$ and will thus definitely determine the direction of the resultant velocity, with the quadrant in which it lies.

2·9. Triangle of velocities.

If a moving point possesses simultaneously two velocities represented in magnitude, direction and sense successively by the two sides of a triangle taken in order, their resultant is represented by the third side in opposite order.

Let AB and BC represent in magnitude, direction and sense, the two simultaneous velocities of a point. Complete the parallelogram $ABCD$. Then AD, being equal and parallel

to BC, represent so far as magnitude, direction and sense are concerned, the same velocity as is represented by BC. Hence, the two simultaneous velocities of the moving point, being represented in magnitude and direction by AB and AD, are equivalent, by parallelogram of velocities to the resultant velocity represented by AC.

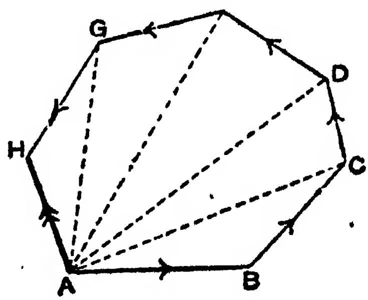


Note. In vector notation, $\overline{AB}+\overline{BC}=\overline{AC}$.

2·10. Polygon of velocities.

If a moving point possesses several simultaneous velocities, represented in magnitude, direction, and sense successively by a series of lines joined end to end, in the same order, then the line drawn to close up the polygon so formed, in reverse order, will represent its resultant velocity.

Let the several simultaneous velocities possessed by a particle be represented in magnitude, direction, and sense successively by the lines AB, BC, CD,..., GH. Then the resultant velocity of the particle will be represented by the straight line AH in magnitude, direction and sense.



Now the resultant of the velocities represented by AB and BC is, by triangle of velocities, represented by AC. Again, the resultant of the two velocities represented by AC and CD is represented by AD. Hence, AD represents the resultant of the three simultaneous velocities represented by AB, BC, CD. Proceeding in this manner, AH will represent the resultant of the simultaneous velocities represented by AB, BC, CD,..., GH.

Cor. If a point possesses simultaneously velocities which are represented in magnitude, direction and sense by the sides taken in order of a closed polygon, then the point will remain, on the whole, at rest.

2·11. Illustrative Examples.

Ex. 1. *A point possesses two simultaneous velocities, one $7\frac{1}{2}$ miles per hour towards the East, and the other 10 feet per sec. at an angle 60° North of East. Find the magnitude and direction of a third velocity which must be imparted to it so that it may remain at rest.*

$$7\tfrac{1}{2} \text{ miles per hour} = \frac{15}{2} \times \frac{1760 \times 3}{60 \times 60} \text{ feet per second} = 11 \text{ ft. per sec.}$$

Now, the resultant of the two given velocities 11 ft./sec. towards the East and 10 ft./sec. at an angle 60° North of East is

$$w = \sqrt{11^2 + 10^2 + 2.11.10 \cos 60^\circ}$$

$$= \sqrt{121 + 100 + 110} = \sqrt{331} \text{ ft./sec.}$$

in a direction making an angle

$$\theta = \tan^{-1} \frac{10 \sin 60^\circ}{11 + 10 \cos 60^\circ} = \tan^{-1} \frac{10 \frac{\sqrt{3}}{2}}{11 + 10.\frac{1}{2}}$$

$$= \tan^{-1} \frac{5\sqrt{3}}{16} \text{ North of East.}$$

In order that the particle may remain at rest, a velocity equal and opposite to this resultant velocity must be imparted to it *i.e.*, a velocity $\sqrt{331}$ ft./sec. in a direction making an angle $\tan^{-1} \frac{5\sqrt{3}}{16}$ South of West.

Ex. 2. *A swimmer can swim in still water at the rate of 4 miles per hour. He wishes to cross a river flowing along a straight course at the rate of 2 miles per hour, so as to reach the directly opposite point on the other bank. In what direction should he attempt to swim?*

If he wishes to cross the river in shortest time, what direction should he take to swim?

Let AB represent the velocity of the current, and AC that of the swimmer, at angle α with the current. Due to these two simultaneous

velocities, the resultant velocity of the swimmer is by parallelogram of velocities represented by the diagonal AD.

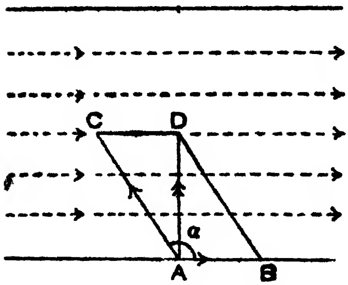


Fig. (i)

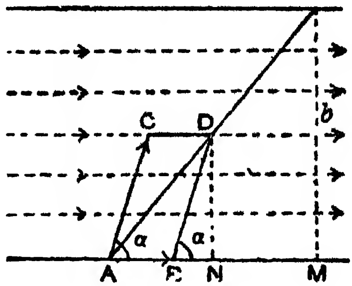


Fig. (ii)

(i) Now if this resultant AD be perpendicular to AB (*Fig. i*),

we have $\cos\ (180° - \alpha) = \cos ABD = \dfrac{AB}{BD} = \dfrac{AB}{AC} = \dfrac{2}{4} = \dfrac{1}{2} = \cos 60°$.

$\therefore\ \ 180° - \alpha = 60°$ or $\alpha = 120°$.

Hence, the swimmer should swim at an angle 120° with the direction of the current in order to cross the river perpendicularly, so as to reach the directly opposite point E on the other bank.

(ii) If the man wishes to cross the river in shortest time, we note that in a unit time, as he reaches the point D with the resultant velocity AD (*Fig. ii*), the actual breadth of the river crossed over by him is $DN = BD \sin DBN = AC \sin \alpha = 4 \sin \alpha$. Hence, b denoting the total breadth of the river, the time taken to cross the river $= \dfrac{b}{4 \sin \alpha}$.

[*Alternatively*, we may note that AD representing the resultant velocity, the man reaches the opposite bank at E, for which the time is

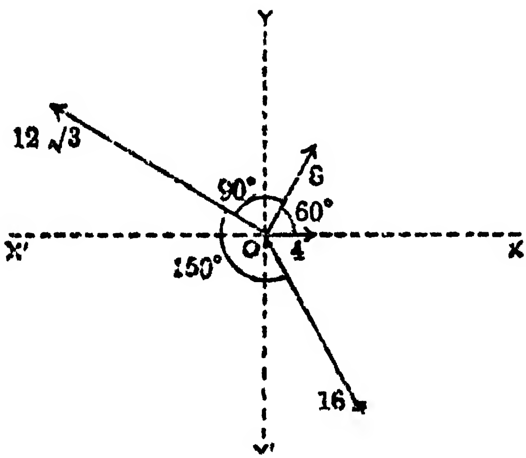
$\dfrac{AE}{AD} = \dfrac{EM}{DN}$ (from similar triangles, EM being the breadth of the river) $= \dfrac{b}{4 \sin \alpha}$.]

Now, this is least when $\sin \alpha$ is greatest, namely 1, which requires $\alpha = 90°$.

Thus to cross the river in shortest time, the swimmer should take to swim in a direction perpendicularly to the current. (His resultant motion however is not perpendicular to the current in this case.)

Note. The time of crossing the river as seen above, is independent of the velocity of the current.

Ex. 3. *A point possesses four simultaneous velocities whose magnitudes are 4 ft./sec., 8 ft./sec., 12 $\sqrt{3}$ ft./sec., and 16 ft./sec. respectively. The angle between the directions of the first and the second is 60°, between the second and the third 90°, and between the third and the fourth 150°. Find the magnitude and direction of the resultant velocity.*



Let us take OX coincident with the direction of the first velocity, and OY perpendicular to it. Clearly, the angles made by the four given velocities with OX, all measured positively, are respectively, 0°, 60°, 150° and 300°. Now resolving each velocity along the two directions OX and OY,

the algebraic sum of the resolved parts along OX

$$= 4 + 8 \cos 60° + 12\sqrt{3} \cos 150° + 16 \cos 300°$$

$$= 4 + 8.\tfrac{1}{2} + (12\sqrt{3}).(-\tfrac{1}{2}\sqrt{3}) + 16.\tfrac{1}{2} = -2 \text{ ft./sec.}$$

and the algebraic sum of the resolved parts along OY

$$= 4 \sin 0° + 8 \sin 60° + 12\sqrt{3} \sin 150° + 16 \sin 300°$$

$$= 0 + 8.\tfrac{1}{2}\sqrt{3} + 12\sqrt{3}.\tfrac{1}{2} + 16.(-\tfrac{1}{2}\sqrt{3})$$

$$= 2\sqrt{3} \text{ ft./sec.}$$

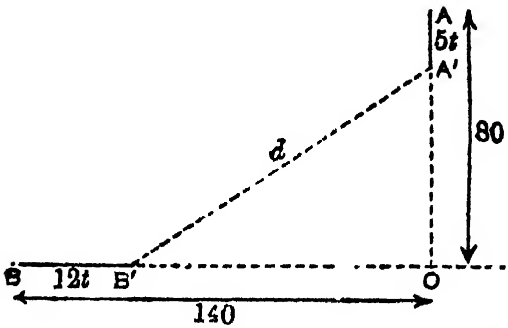
Thus, if v be the resultant velocity in a direction making an angle θ with OX,

$$v\cos\theta=-2,\quad v\sin\theta=2\sqrt{3},\quad\text{whence } v^2=16 \text{ or } v=4,$$

$$\therefore\ \cos\theta=-\frac{1}{2},\quad \sin\theta=\frac{\sqrt{3}}{2}\cdot\ \text{giving } \theta=120°.$$

Hence, the resultant velocity of the particle is 4 ft./sec. in a direction making an angle 120° with the first velocity.

Ex. 4. *A bus is moving at 12 miles per hour along a straight road, and a man, running at 5 miles per hour along a perpendicular road, sees it, when it is 140 yds. short of the junction, and he himself is 80 yds. short. Show that he can never get nearer to the bus than 20 yds.*



The bus B is moving along BO, and the man A runs along AO, A sees B when $BO=140$ yds. $=\frac{7}{88}$ miles, and $AO=80$ yds. $=\frac{1}{22}$ miles.

After any time, t hours say, the distances moved over by the bus and the man are respectively $BB'=12t$ miles, and $AA'=5t$ miles.

Hence, the distance $A'B'$ $(=d)$ between them is given by

$$\begin{aligned}d^2&=\left(\tfrac{7}{88}-12t\right)^2+\left(\tfrac{1}{22}-5t\right)^2\\&=169t^2-\frac{26}{11}t+\frac{65}{88^2}\\&=\left(13t-\frac{1}{11}\right)^2+\left(\frac{65}{88^2}-\frac{1}{11^2}\right)\\&=\left(13t-\frac{1}{11}\right)^2+\frac{1}{88^2}.\end{aligned}$$

Now the variable portion involving t being a perfect square, can never be negative, its least value being zero. Hence, the least value

of $d^2 = \frac{1}{88^2}$, *i.e.*, the shortest possible distance of the man from the bus is $\frac{1}{88}$ miles = 20 yds.

Note. For an alternative method by using relative velocity, see Ex. 8, § 3·6.

Examples on Chapter II

1. A man walks towards the East a distance of 3 miles at the rate of 5 miles per hour, and then walks towards the North a distance of 4 miles at the rate of 3 miles an hour. Find the average speed and the average velocity of the man for the whole journey.

2. A point moves in a straight line with a velocity of 3 feet per second. After 3 seconds, it has an additional velocity of 4 feet per second at right angles to its original direction of motion. Find its distance from the starting point 2 seconds after this.

3. A thief, when detected, jumps out of a running train at right angles to its direction with a velocity of 5 ft. per sec. If the velocity of the train be 30 miles per hour, find in which direction the thief falls.

4. What velocity must be communicated by the bat to a cricket ball travelling horizontally in the line of wickets at 90 ft. per second, so as to make it travel at right angles to its original path with a speed of 120 ft. per sec.?

5. A point which possesses velocities of 7, 8 and 13 ft. per second in different directions is at rest. Find the angle between the directions of the two smaller velocities.

6. Three velocities whose ratios are $(\sqrt{3}+1) : \sqrt{6} : 2$ are simultaneously impressed on a particle and it is noticed that the particle does not move. Find the angles at which the directions of the velocities are inclined to each other.

7. A particle P has simultaneously three velocities represented by PA, PB, PC where A, B, C are three non-collinear fixed points. Where should P be situated so that the particle may remain at rest?

8. A man rows his boat across a river 100 yds. wide, always pointing his boat up-stream at an angle of 30° with the bank. How long does he take to cross the river if he rows with a velocity of 6 ft. per sec., and current flows at the rate of 3 miles per hour? Find how far down the river the boat will reach the opposite bank below the point at which it was originally directed.

9. A man rows directly across a flowing river in time t_1, and rows an equal distance down the stream in time t_2. If u be the speed of the man in still water and v that of the stream, show that

$$t_1 : t_2 = \sqrt{u+v} : \sqrt{u-v}.$$

10. Two boats each moving with a velocity of 5 miles per hour try to cross a stream of breadth 880 yds., running with a velocity of 3 miles per hour. One boat crosses the stream by the shortest path and the other in the shortest time. If they start together, find the interval between their times of arriving at the opposite bank.

11. A river of breadth one mile has a current flowing at the rate of 2 miles per hour. A swimmer who can swim at the rate of 4 miles per hour in still water wishes to reach the directly opposite point on the other bank, but choosing a wrong direction to swim, reaches the opposite bank ½ mile down the desired point. Find the deviation of the chosen direction from the right one.

12. A man can swim directly across a stream of breadth 100 yds. in 4 minutes when there is no current and in 5 minutes when there is current. Find the velocity of the current.

13. Two motor-boats start simultaneously from two points A and B, the first one moving with a uniform velocity of $10\sqrt{3}$ miles per hour in a direction making an angle of 30° with AB. Find the direction in which the second should move uniformly with a velocity of 10 miles per hour so that it may meet the first one.

If the second boat moves at an angle of 45° with AB, with what velocity should it go in order that it may meet the first?

14. Two cyclists P and Q are respectively at points A and B which are $\sqrt{3}+1$ miles apart on a field. P rides away with a uniform velocity of $5\sqrt{2}$ miles per hour in a direction making an angle 45° with AB. Q starts at the same instant to move with a uniform velocity of 10 miles per hour and catches P. Find the time that elapses from start before Q catches P.

15. A point has equal velocities in two given directions; if one of these velocities be halved, the angle which the resultant makes with the other is halved also. Find the angle between the given directions.

16. A train is moving with a uniform velocity v along a straight railway line and a motor-car runs on a parallel road in the same direction, the distance between the road and the railway line being a. A passenger of the train observes the car to be always in a line with a fixed tree whose distance from the railway line is $b\,(b > a)$. Prove that the velocity of the car is uniform and find its magnitude.

17. An aeroplane, travelling in still air at the rate of 125 miles per hour, starts from a point P to reach a point Q due North of it, 300 miles away. There is a wind blowing due West at the rate of 35 miles per hour, but when half the distance has been covered, the velocity of the wind increases to 75 miles per hour, and the aeroplane adjusts its head accordingly so that it continues its course along PQ as before, and reaches Q. Find the time taken over the flight.

18. A destroyer steaming North at the rate of 15 miles per hour observes a sea-plane carrier due East of itself at a distance of 10 miles, the latter steaming due West at the rate of 20 miles per hour. After what time are they at the least distance from one another, and what is this least distance ?

19. Two straight railway lines meet at right angles. A train starts from the junction along one line, and at the same instant, another train starts towards the junction from a station on the other line, and they move at the same uniform speed. Show that they are nearest to each other when they are equally distant from the junction.

20. A battleship leaves a certain port and steams N. W. at 15 knots. Another ship leaves the same port at the same instant and steams W. S. W. at 12 knots. Their wireless instruments are capable of communication up to 500 nautical miles. How long may the ships expect to remain in touch ?

[1 *knot* = 1 *nautical mile per hour*]

21. A particle possesses simultaneously three velocities, u, v, w in directions inclined at angles α, β, γ with one another ; show that the resultant velocity is

$$[u^2 + v^2 + w^2 + 2uv \cos \alpha + 2vw \cos \beta + 2wu \cos \gamma]^{\frac{1}{2}}.$$

22. A point possesses five simultaneous velocities 10, 20, 30, 40 and 50 ft. per sec. respectively. The first three are respectively towards E., N. E. and S. S. W. The fourth is 15° West of North, and the fifth 30° East of South. Find the displacement of the particle 5 secs. after start.

ANSWERS

1. $3\frac{12}{13}$ m. p. h. : $2\frac{17}{25}$ m. p. h. at an angle $\tan^{-1} \frac{4}{3}$ North of East.
2. 17 ft. **3.** At an angle $\tan^{-1} \frac{4}{7}$ with the direction of motion of the train. **4.** 150 ft./sec. at an angle $\cos^{-1} (-\frac{4}{5})$ with the original path. **5.** 60°. **6.** 135°, 105°, 120°.
7. At the centroid of the triangle ABC. **8** $1\frac{2}{3}$ min. ; 440 ft.
10. $1\frac{1}{2}$ min. **11.** 30°. **12.** 15 yds./min.
13. 60° with BA ; $5\sqrt{6}$ miles/hr. **14.** 12 min.
15. 120°. **16.** $\left(1 - \frac{a}{b}\right)v$. **17.** 2 hrs. 45 min.
18. $19\frac{1}{5}$ min. ; 6 miles. **20.** 32·9 hrs.
22. 164 ft. nearly, in a direction approximately at an angle $\tan^{-1} \frac{2}{3}$ S. of E.

CHAPTER III

RELATIVE VELOCITY

3·1. In the previous chapter we have defined velocity of a moving point P as its rate of change of *position*. Now to define this position we must have some frame of reference, or a point of reference, say O, with respect to which the position of P at any instant is given by the straight line joining O to P. When this straight line OP alters, either in length or in direction, or in both, we say that P has changed its position, and has thus moved, as seen from O. If we say that the point of reference O is also moving, we must have some other frame of reference in mind, with respect to which O is changing its position. In fact, we have no idea, nor can define, what absolute motion of a point would mean and every motion in that sense is relative, that is with reference to some contemplated observer.

Every one must have noticed from a moving railway train that trees or telegraph posts outside seem to approach and then move rapidly backwards. Really the distance of the tree from the observer is changing though not due to any movement of the tree, and this change of position of the tree gives an idea of a motion of the tree to the observer. But as he is conscious that the tree is not capable of moving on the ground, he attributes this apparent motion of the tree to his own motion with the running train.

Usually we speak of a body to be at rest when it does not change its position with respect to surrounding objects on the surface of the earth, and say that it is in motion when it changes its position with respect to the so-called *fixed* objects on the surface of the earth. But we also seem to know that the earth is not fixed, and that it moves round the sun at a speed of nearly 19 miles per second. In this latter description our contemplated observer is at the sun. Again, the sun is described to be moving with the whole solar system towards a so-called *fixed* star, which in this case is taken as the observer for reference. In fact, it

should be borne in mind, as stated before, that absolute motion (or true motion as we may be tempted to call it), or rest, without any reference to any observer, is perfectly meaningless, from the very definition of the terms.

In this book, unless anything is mentioned, we shall speak of *true motion* or *rest* by considering the points of reference to be the so-called fixed objects on the surface of the earth*. We then proceed to define relative velocity, and consider theorems in that connection as given below.

3·2. Relative velocity.

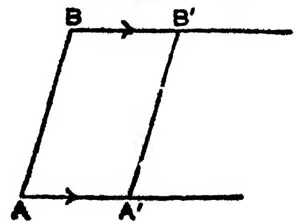
When two points A and B, which may be both moving on the surface of the earth, are considered, the rate of change of position of B as seen from A, (this position being indicated by the line joining A to B) is defined as the relative velocity of B with respect to A.

This relative velocity of B with respect to A is obtained by compounding with the velocity of B, a velocity equal and opposite to that of the observer A as proved in the next article.

3·3. Determination of relative velocity.

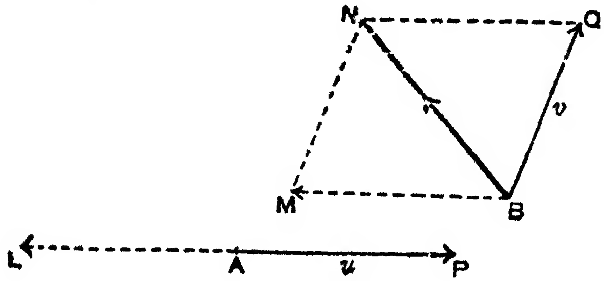
The relative velocity of one moving point B with respect to another moving point A is obtained by compounding with the given velocity of B, a velocity equal and opposite to that of the observer A.

First of all, let A and B both move with equal velocities in parallel directions. In any time, their displacements AA' and BB' are equal and parallel, and hence the line joining them remains equal and parallel to itself. Thus, the distance and direction of the line joining A to B *i.e.*, the position of B



*In books on Astronomy other points and frames of reference are used for describing motion according to the circumstances.

as observed from A remains unaltered. Thus, B will appear to be at rest, and their relative velocity with respect to each other is nil.



Next suppose A and B move with any velocities u and v represented by AP and BQ respectively. To both A and B apply equal and parallel velocities represented by AL and BM, each equal and opposite to AP. These equal and parallel velocities of A and B produce no relative motion between them, and so the apparent motion of B as seen from A remains unaltered by this addition. A is now brought to rest, and the resultant motion of B is given by the diagonal BN, which thus represents the required relative velocity of B as it appears to the observer A.

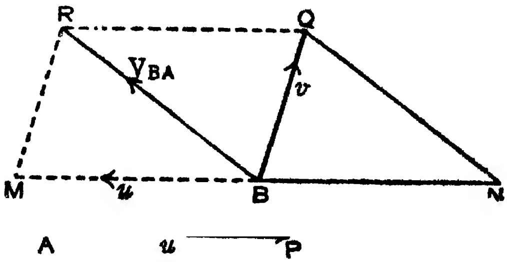
In vector notation, $\overline{\mathbf{V}}_{BA} = \overline{\mathbf{V}}_B - \overline{\mathbf{V}}_A$.

3·4. Determination of true velocity when the apparent velocity is given.

When the relative velocity of a point B with respect to an observer A is given, as also the velocity of the observer A, the true velocity of B is obtained by compounding these two given velocities.

As the relative velocity of B with respect to A is found as the resultant of the true velocity of B and the velocity of A reversed, if we compound with it the velocity of A, the two latter velocities will neutralise each other, leaving as the resultant, the true velocity of B.

Geometrically, BR representing the given relative velocity V_{BA} of B with respect to A, and AP representing the given velocity u of A, if we draw BM equal and parallel to AP in opposite sense, and complete the parallelogram



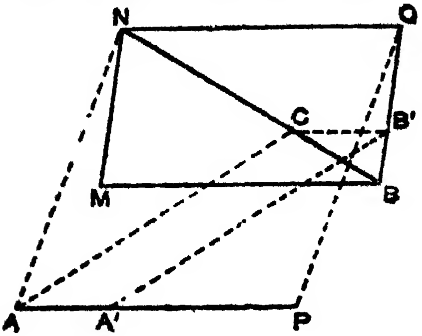
$BMRQ$ with BR as diagonal and BM as side, then the other side BQ represents the true velocity of B, for this compounded with BM, the reversed velocity of A, gives as the resultant, the relative velocity BR in question. Now, if MB be produced to N making $BN = BM$, the figure $BRQN$ is evidently a parallelogram, in which BQ, the required true velocity of B is the diagonal with BR and BN as adjacent sides. As BN is equal and parallel to AP, it follows that the required true velocity BQ of B may as well be determined by combining with the given relative velocity BR, a velocity BN, equal and parallel to that of the observer A in the *same* sense.

In vector notation, $\overline{V}_B = \overline{V}_{BA} + \overline{V}_A$.

3·5. It may be noted that the actual way in which the distance between two points which are both moving, or the direction of one as seen from the other at different instants alters, may be estimated either by considering the actual motions of both the points during the interval, or equally well, by assuming one to be at rest and making the other move with relative motion, and this second method is simpler in actual practice.

Thus, if AP and BQ represent the actual velocities of A and B, after unit time the distance between them is PQ. Now, keeping A fixed, and assuming B to move with the relative velocity BN, it is clear from the mode of construct-

ing relative velocity, that QN is equal and parallel to AP, and so AN is equal and parallel to PQ. AN thus gives the distance and direction between the two points after unit time equally well as PQ. At any other instant, say, after time τ from start, A moves to A' and B to B' where $AA' = \tau\, AP$ and $BB' = \tau.BQ$, and the distance between the points then is $A'B'$. But keeping A fixed, if we assume B to move with relative velocity BN, it reaches the point C after time τ where $BC = \tau.BN$ and the distance between the moving points thus estimated is given at the instant by AC. Now $\dfrac{BC}{BN} = \tau = \dfrac{BB'}{BQ}$. Therefore $B'C$ is parallel to QN and $= \tau.QN$, *i.e.*, $= \tau.AP$. Hence, $B'C$ is equal and parallel to $A'A$ and so AC is equal and parallel to $A'B'$.



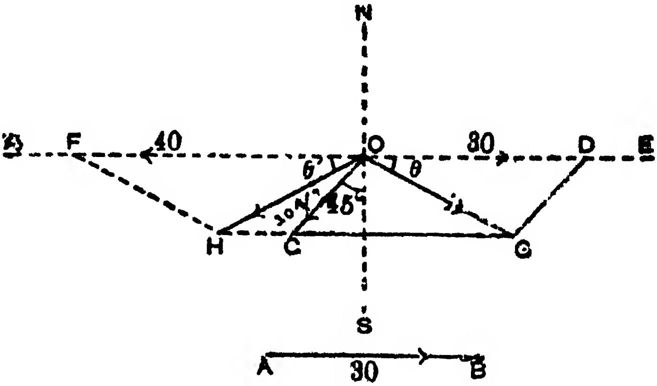
In problems therefore, where we are *to determine the least distance between two points, which are both moving* with given velocities, it is always advisable to reduce one to rest and make the other move with relative velocity, and then estimate the least distance, which will give us the result, same as the actual result. [*Cf. Ex. 3* § *3·6*]

3·6. Illustrative Examples.

Ex. 1. *To a passenger on a train running due East at the rate of 30 miles per hour, wind appears to blow from N. E. at the rate of 10 $\sqrt{2}$ miles per hour ; find the true velocity of the wind. Find also its apparent direction, when the speed of the train increases to 40 miles per hour.*

AB representing the velocity of the train, (30 miles/hr. due East) it also represents the true velocity of the observer. Let OC represent the apparent velocity of the wind (10 $\sqrt{2}$ m./hr. from N. E. towards S. W.).

Combining with OC, a velocity OD equal and parallel to AB in the same direction, the resultant OG represent the true velocity of the wind.



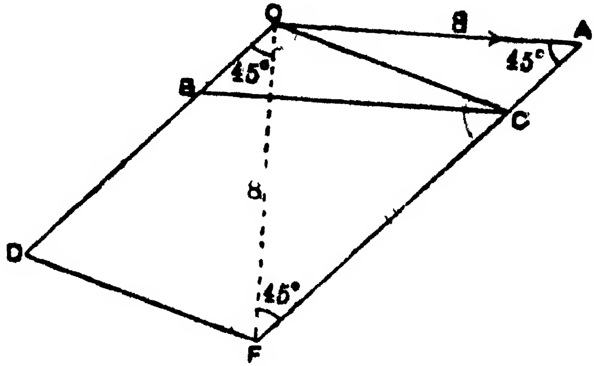
Now, resolving OC into its components $10\sqrt{2}\cos 45^\circ$ and $10\sqrt{2}\sin 45^\circ$, *i.e.*, 10 and 10 towards West and South respectively, and combining with OD (30 towards the East), we get $30-10=20$ m/h towards the East, and 10 m/h towards the South, the resultant of which is $OG=\sqrt{20^2+10^2}=10\sqrt{5}$ miles/hr., at an angle $\theta=\tan^{-1}\frac{10}{20}$, *i.e.*, $\tan^{-1}\frac{1}{2}$ S of E., giving the true velocity of the wind in magnitude and direction.

[*Alternatively*, noting that angle $DOC=135^\circ$, we might apply the mathematical formulæ of § 2·5 to get the magnitude and direction of the resultant OG.]

When the speed of the train increases to 40 miles/hr., to get the apparent direction of wind, we combine with its true velocity OG, a velocity OF, equal and opposite to that of the observer (*i.e.*, 40 towards the West) and find the resultant OH. Now OG, as proved before, being equivalent to 20 m/h towards the East and 10 m/h. towards the South, we get OH to be the resultant of $40-20=20$ m/h towards the West and 10 m/h towards the South. Thus, if $\angle HOF=\theta'$, $\tan\theta'=\frac{10}{20}=\frac{1}{2}$, or $\theta'=\tan^{-1}\frac{1}{2}$.

Hence, the apparent direction of the wind now is at an angle $\tan^{-1}\frac{1}{2}$ South of west.

Ex. 2. *To a cyclist travelling at 8 miles per hour due East, the wind appears to come from the North-East; but when he travels North-East at the same speed, it appears to come from the North. Find the true direction and velocity of the wind.* [*C. U. 1941*]



Let OB represent the apparent velocity of the wind from N. E. when the cyclist is travelling at 8 m/h due East. Combining with OB the velocity OA of the cyclist in the same direction (towards the East) and completing the parallelogram $OBCA$, the resultant OC represents the true velocity of the wind.

When the cyclist is moving with the same speed towards North-East, combining this velocity reversed (represented by OD) with OC, and completing the parallelogram $OCFD$, the resultant OF represents the apparent velocity of the wind then, and this is given to be from the North, as shown in the above figure.

Now, in the figure $\angle BOF = 45°$; $\therefore$ $\angle OFA = 45°$,

and as $\angle AOF = 90°$, $\angle OAF = 45°$.

Thus, $OF = OA = 8$. Also $CF = OD = 8$.

$\therefore$ $\angle FOC = \angle FCO = \frac{1}{2}(180° - 45°) = 67\frac{1}{2}°$.

$\therefore$ $\angle AOC = 22\frac{1}{2}°$.

Also, from ΔFOC, $OC = \sqrt{FO^2 + FC^2 - 2FO.FC \cos OFC}$

$= \sqrt{8^2 + 8^2 - 2.8.8 \cos 45°} = 8\sqrt{2 - \sqrt{2}}.$

Thus, the true velocity of the wind is $8\sqrt{2-\sqrt{2}}$ miles per hour at angle $22\frac{1}{2}°$ S. of E., towards E. S. E. (from W. N. W.).

Alternatively, from $\triangle OCF$,

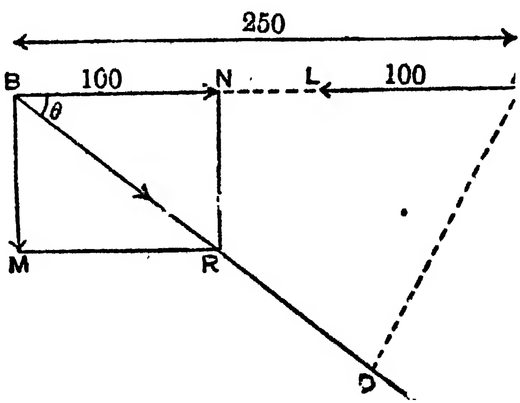
$$\frac{OC}{\sin 45°}=\frac{8}{\sin 67\frac{1}{2}°}=\frac{8}{\cos 22\frac{1}{2}°}$$

$$=\frac{8}{\sqrt{\frac{1}{2}(1+\cos 45°)}}=\frac{8}{\sqrt{\frac{1}{2}\left(1+\frac{1}{\sqrt{2}}\right)}}$$

$$=\frac{8}{\frac{1}{2}\sqrt{2+\sqrt{2}}}=\frac{8\sqrt{(2-\sqrt{2})}}{1/\sqrt{2}}.$$

Hence, $OC=8\sqrt{2}\sqrt{2-\sqrt{2}}$.

Ex. 3. *At a particular instant two aeroplanes are at a distance of* 250 *miles, one due East of the other. The first one is moving westwards at the rate of* 100 *miles per hour, and the second with a velocity of* 75 *miles per hour towards the South. Find the time when they are nearer to each other, and the least distance between them.*



A and *B* represent the positions of the aeroplanes at the given instant, and *AL* and *BM* their velocities. To find when they are nearest, or the least distance between them, as explained in § 3·5, we may keep one, (say *A*), fixed and make the other move with the relative velocity *BR*, as shown in the above figure. Clearly $BR=\sqrt{100^2+75^2}=125$ m/hr.

AD being now drawn perpendicular to BR, the least distance of B from A is AD.

Now $AD = AB \sin\theta$ (where $\theta = \angle ABD$)

$$= AB \cdot \frac{NR}{BR} = 250 \times \frac{75}{125} = 150 \text{ miles.}$$

Also, $BD = AB \cos\theta = 250 \times \frac{100}{125} = 200$ miles, and with relative velocity BR *i.e.*, 125 m/h., the time taken to travel this distance BD relatively is $\frac{200}{125}$ hrs. = 1 hr. 36 m.

Thus, the two aeroplanes will be nearest to each other 1 hr. 36 m. from the given instant, and the least distance between the two is 150 miles.

Note. For an alternative method, *Cf.* Ex. 4, § 2·11.

Examples on Chapter III

1. One boat is sailing due North at the rate of 12 miles per hour and another boat is sailing North-West at the rate of $12\sqrt{2}$ miles per hour. Find in magnitude and direction, the velocity of the second boat relative to the first.

2. A schoolboy holding an umbrella runs with a velocity equal to that of the rain falling vertically, in consequence of which the rain strikes him in the face. At what angle should he hold the umbrella in order to protect him best ?

3. On a rainy day when a man is walking at the rate of 4 miles an hour, he is struck by the rain vertically, and when he increases his velocity to 8 miles an hour, the rain strikes him at an angle of 45°. Find the magnitude and direction of the velocity of the rain.

4. To a man walking at the rate of 3 miles an hour, rain appears to fall vertically ; if he increases his speed to 5 miles an hour, it appears to fall at an angle of 30° with the vertical. Find the actual direction and velocity of the rain. [*U. P. 1937*]

5. A train is travelling N. at 60 m.p h. and the wind is blowing from the S. W. at 20 m.p.h. Find the direction of the trail of the smoke of the engine.

[Assume that the smoke loses the velocity of the train as soon as it leaves the funnel and moves with the velocity of the wind.]

6. A steamer is going due N, with velocity v, the smoke from the chimney points θ degrees S. of E. If the wind be coming from due West, find its velocity.

7. Three points P, Q, R move with the same velocity v, along the sides BC, CA, AB respectively of an equilateral triangle. Find the velocity of any one relative to any other in magnitude and direction.

8. (i) Two points P and Q are moving with velocities u and v respectively along two straight lines inclined at an angle α. Find the magnitude and direction of the relative velocity of Q with respect to P in terms of u, v, α. Interpret the result when $\alpha=0$ and $\alpha=\pi$. [*C. U. 1949*]

(ii) Two particles move with speeds u and v respectively in opposite sense along the circumference of a circle. In what positions will their relative velocity be greatest, and least, and what are its values then ?

What would happen if they move in the same sense ?

9. Given the relative velocity of A with respect to B and also the relative velocity of B with respect to C, show how you will proceed to determine the relative velocity of C with respect to A.

10.* Two trains whose lengths are respectively 130 and 110 ft. are moving in opposite directions on parallel lines, the velocity of the first being double that of the second. They are observed to pass each other completely in 4 secs. Find the velocity of each train.

11. A bomber moves due East at 100 m.p.h. over a town X at a certain time. Six minutes later a pursuit plane starts from a station Y which is 40 miles due South of X and flies North-East. If both maintain their course, find the velocity with which the pursuit plane must fly in order to overtake the bomber.

12.* A person travels due East at the rate of 4 miles per hour and observes that the wind seems to blow directly from the North ; he then doubles his speed and the wind appears to come from the North-East. Determine the direction and velocity of wind. [*U. P. 1940* ; *C. U. 1943*]

13. A person travelling towards the North-East finds that the wind appears to blow from the North, but when he doubles his speed, it seems to come from a direction making an angle $\tan^{-1}\frac{1}{2}$ East of North. Find the true direction of the wind.

14. A steamer is travelling due East at the rate of u miles an hour. A second steamer is travelling at $2u$ miles an hour in a direction θ North of East, and appears to be travelling North-East to a passenger on the first steamer. Prove that

$$\theta = \tfrac{1}{2}\sin^{-1}\tfrac{3}{4}.$$ [*C. U. 1945 ; '49*]

15. A river is flowing from West to East at 2 miles per hour, and a boat is rowed with a velocity of 4 miles per hour due South relative to the current. A hackney carriage runs on a road parallel to the river towards the West at the rate of 6 miles per hour. Find the apparent velocity of the boat as seen by an observer on the carriage.

16. Two railway lines cross at right angles. One train running at a speed of 30 miles per hour along one line crosses the junction at 7 p.m. Another train moving along the other line at 40 miles per hour crosses the junction at 12 midnight. Find the time when they were nearest to each other.

17. (i) Two roads cross at an angle of 60°. Two persons, one on each road walking at the same speed, are approaching the crossing (acute angle), their simultaneous distances being 100 yards and 200 yards respectively. Find their distances from the crossing at the instant when they are nearest to one another. [*U. P. 1924*]

(ii) Two particles start simultaneously from the same point and move along two straight lines inclined at an angle α, one with uniform velocity u and the other from rest with uniform acceleration f. Show that their relative velocity is least after a time $u\cos\alpha/f$ and that the least relative velocity is $u\sin\alpha$.

18. To an observer on a train moving at 30 miles per hour due North, wind appears to blow from 15° E. of N., and from a motor-car running at $15(\sqrt{3}-1)$ miles per hour due East, it appears to come from 15° N. of E. Find the true direction of wind.

19. A pistol shot is fired on a running train at an angle α with its direction of motion. The shot enters a carriage at a corner furthest from the engine and passes out at the diagonally opposite corner. If u be the velocity of the train in miles per hour and a and b are the length and breadth of the carriage in feet, show that the time the shot takes to pass through the carriage is $15(b \cot \alpha - a)/22u$ seconds.

20. A battleship leaves a port P, and sails northwards at the rate of 20 miles per hour. A submarine simultaneously starts from a point Q, 50 miles East of P, with a uniform velocity of $10\sqrt{6}$ miles per hour with the intention of having the ship within 25 miles range of itself. Find the extreme directions within which it must direct its motion.

ANSWERS

1. 12 m.p.h. westwards.
2. 45° with the vertical.

3. $4\sqrt{2}$ m.p.h. at 45° with the vertical on the side towards which the man walks.

4. $\sqrt{21}$ m/h at $\tan^{-1}\frac{\sqrt{3}}{2}$ with the vertical in the forward sense.

5. $\cot^{-1}(3\sqrt{2}-1)$ *i.e.*, 17° 8′ East of South.

6. $v \cot \theta$.
7. $v\sqrt{5}$ parallel to altitude.

8. (i) $\sqrt{u^2+v^2-2uv\cos\alpha}$ at an angle $\tan^{-1}\dfrac{v\sin\alpha}{v\cos\alpha-u}$ with the direction of motion of P.

When $\alpha=0$, P and Q move in the same direction, and their relative velocity $=u\sim v$. When $\alpha=\pi$, they move in opposite directions, and the relative velocity $=u+v$.

(ii) When moving in opposite sense, greatest relative velocity is $u+v$ when they meet, and least is $u\sim v$ when they are diametrically opposite. When moving in the same sense, greatest relative velocity is $u+v$ when diametrically opposite, least $u\sim v$ when they meet.

9. Combine the two given relative velocities and reverse the direction of the resultant.

10. $27\frac{3}{11}$ and $13\frac{7}{11}$ m.p.h.

11. 168 56 m.p.h.

12. $4\sqrt{2}$ m.p.h. from N.W.

13. Towards the East.

15. $4\sqrt{5}$ m/hr. at $\tan^{-1}\frac{1}{2}$ S. of E.

16. 10 hrs. 12 m. P.M.

17. 50 yds. each.

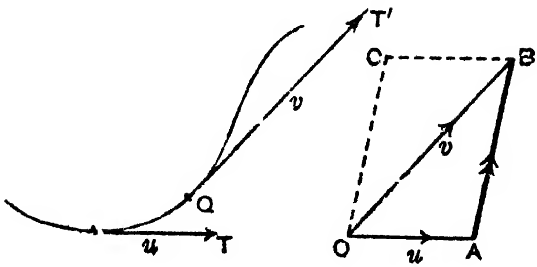
18. From 30° N. of E.

20. 75° N. of W. and 15° N. of W.

CHAPTER IV

ACCELERATION

4·1. Change of Velocity.



Let a particle be moving in any manner. At any two instants separated by any interval of time its positions are P and Q, and its velocities are u and v respectively along the tangents PT and QT' to its path. From any point O let us draw the straight lines OA and OB to represent these velocities in magnitude and direction.

Then the line AB from A to B, joining the extremities of OA and OB represents in magnitude, direction and sense, the change of velocity during the interval.

Complete the parallelogram $OABC$. By parallelogram of velocities, the velocity v, represented by OB, is equivalent to the components OA and OC. Thus, while at P the velocity of the particle was u represented by OA, after a time t, while at Q, its velocity is equivalent to u together with another velocity represented by OC.

Hence, during the interval t, a velocity represented by OC has been added to the original velocity to make up the final velocity. Thus the change of velocity during the interval is represented by OC, or what amounts to the same

thing. by the line AB which is equal and parallel to it in magnitude and direction.

4·2. Acceleration.

The rate of change of velocity of a moving particle is defined to be its acceleration.

Acceleration of a moving point is said to be *uniform* when equal changes of velocity in the same direction take place in equal intervals of time, however small these time-intervals may be taken.

Let us first of all consider the case when *a particle always moves along the same straight line*, but with variable velocity. For instance, let us consider an engine moving along a straight railway line, and suppose its velocity at a particular instant to be 10 miles per hour. Three minutes later, let its velocity be observed to be 28 miles per hour. Then in three minutes, a velocity of 18 miles per hour is added to the original velocity in the same direction. Assuming the rate of increase to be uniform, the acceleration of the engine is 6 miles per hour in each minute, *i.e.*, $\frac{44}{5}$ feet per second per minute in the direction of the given line.

It may be noted as above, that the expression for acceleration involves two units of time, one involved in the statement of the velocity which is being added and the other in the time in which it is added. The two units of time may be different, as in the above illustration, or may be same, for example, the above acceleration may be described also as $\frac{11}{75}$ feet per second per second, or $\frac{11}{75}$ ft./sec^2, or $\frac{11}{75}$ ft.-sec^{-2} (as it is briefly written). In general, in F.P.S. system, an acceleration will be expressed in ft./sec^2 (or ft-sec^{-2}). and in C.G.S. system, in cms./sec^2 (or cms.-sec^{-2}) units.

When the velocity of a particle moving in a straight line increases, the acceleration is positive, and when it decreases, the acceleration is negative. *A negative acceleration is known as* **retardation.**

Next, let us consider the case when initially a particle starts with a velocity in a given direction, but has a uniform

acceleration in a different direction. As the velocity added in any time to the starting velocity gradually changes, and is in a direction different from that of the latter, the angle made by the resultant velocity of the particle with the initial direction of motion continually changes *i.e.*, the resulting motion of the particle will be along a curved path. An example of this case will be found in the motion of a projectile. [See *Chapter VIII.*]

If the acceleration of a moving point be *non-uniform*, it may change either in magnitude, or in direction, or in both. In case when the acceleration of a moving point is variable, the acceleration at any instant may be measured by the ultimate ratio of the change of velocity in an infinitely small time including the instant, to the time, and is in the direction in which this change of velocity takes place in the limit when the interval of time considered is infinitely small. An example of variable acceleration (where it changes in direction only) is in the case of a point moving in a circle with uniform speed. [See *Normal acceleration, Art. 14·1, Chapter XIV.*]

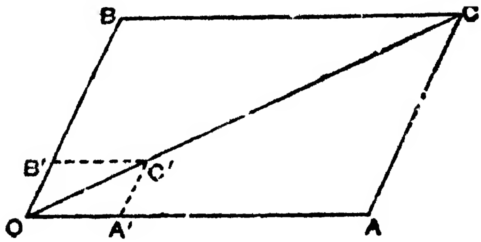
It may be remembered in this connection that the acceleration of a moving point at any instant has got a definite magnitude and direction, and is thus a vector quantity and as such can be represented in magnitude and direction by a straight line, like any other vector quantity.

4·8. Parallelogram of accelerations.

If a moving point possesses simultaneously two uniform accelerations represented in magnitude and direction by the two adjacent sides of a parallelogram meeting at an angular point, they are equivalent to a single resultant accelaration for the moving point, which is represented in magnitude and direction by the diagonal of the parallelogram drawn from that angular point.

The proof depends on the theorem of parallelogram of velocities.

For, let OA and OB represent in magnitude and direction the two simultaneous uniform accelerations of a moving



point. Complete the parallelogram $OACB$ and join the diagonal OC.

Now whatever velocity the particle might have originally, simultaneous velocities represented by OA and OB are added in each unit of time. But these two simultaneous velocities are, by parallelogram of velocities, equivalent to a single velocity represented by OC. Hence, the effect is as if a single velocity OC were added in a unit time to the original velocity of the particle, and OC thus represents the resulting change of velocity of the particle in a unit time. Also this rate of change of velocity must be uniform. For, at any intermediate instant, say after a time τ from the initial moment, due to the two given accelerations the simultaneous changes of velocity of the particle are given by OA' and OB' along OA and OB respectively, where $OA' = \tau.OA$, $OB' = \tau.OB$. Now, if OC' be taken $= \tau.OC$, along OC, then as $\dfrac{OA'}{OA} = \dfrac{OC'}{OC} = \dfrac{OB'}{OB}$, it is easily seen from geometry that $OA'C'B'$ is a parallelogram, and so the two simultaneous velocities OA' and OB' are equivalent to a single velocity represented by OC'. Thus in any time τ the resulting change of velocity is along OC and equal to $\tau.OC$. As this is true whatever τ may be, OC represents the resultant uniform acceleration of the moving point.

4·4. Uniform accelerated motion along a straight line.

The most important case of accelerated motion that we have to consider is, when a particle moves always along the same straight line with a uniform acceleration. In this connection there are three fundamental formulæ which are extremely important. They are as follows :

If a point moves along a straight line with a uniform acceleration f, and if u and v denote its velocities at the beginning and end of any interval of time t considered during its motion, and s the distance covered by it during that time, then

$$\left\{\begin{array}{ll}\text{(i)} & \mathbf{v}=\mathbf{u}+\mathbf{ft} \\ \text{(ii)} & \mathbf{s}=\mathbf{ut}+\tfrac{1}{2}\mathbf{ft}^2 \\ \text{(iii)} & \mathbf{v}^2=\mathbf{u}^2+2\mathbf{fs}.\end{array}\right.$$

The proofs of these formulæ are given below.*

I. *To prove the formula* $\mathbf{v}=\mathbf{u}+\mathbf{ft}$.

u being the initial and v the final velocity corresponding to any interval of time t during the motion of a particle along a straight line with uniform acceleration f, the change of velocity during the interval t is $v-u$, and this change being at a uniform rate f,

$$\frac{v-u}{t}=f,$$

$$\text{or} \qquad v=u+ft.$$

II. *To prove the formula* $\mathbf{s}=\mathbf{ut}+\tfrac{1}{2}\mathbf{ft}^2$.

Let a particle move along a straight line with uniform acceleration f, and let s be the distance described by it in any interval of time t during its motion, u being the velocity at the beginning, and v, that at the end of this interval.

As the velocity gradually changes from u to v, the average velocity during the interval is something intermediate

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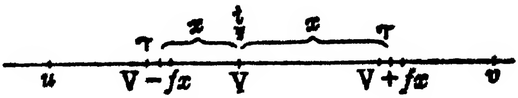
* For alternative method of proof by the application of Calculus, see § 4·7*B*.

between u and v. Let V denote its velocity at the middle of the interval, *i.e.*, at time $\frac{t}{2}$, so that

$$V = u + f\frac{t}{2} \quad \dots \quad \dots \quad \text{(i)}$$



x seconds before the middle instant, the velocity is evidently $V - fx$ (f being the uniform rate at which the velocity increases), and in an extremely small interval of time τ there, the distance travelled by the particle is practically $(V - fx)\tau$.

x seconds after the middle instant, the velocity is $V + fx$, and in an equal small interval τ here, the distance travelled is ultimately $(V + fx)\tau$.

Thus, the total distance described during these two equal small intervals τ, τ is

$$(V - fx)\tau + (V + fx)\tau = 2V\tau,$$

and is the same as if the velocity were V during both these intervals.

As the whole time t can be divided into such pairs of equal small intervals equidistant from the middle instant, and as for each such pair the above conclusion holds, the actual distance travelled during the whole interval t is the same as if the velocity were V from beginning to end. V, therefore, represents the average velocity during the interval.

Hence, $\quad s = Vt = \left(u + f\frac{t}{2}\right)t = ut + \frac{1}{2}ft^2.$

Alternative proof.

Let a particle move along a straight line with uniform acceleration f, and let s be the distance described by it in any interval of time t during its motion, u being the initial and v the final velocity during this interval.

Let us divide the whole time t into n equal parts, each equal to $\frac{t}{n}$.

The velocities of the particle at the beginning of these successive intervals are clearly

$$u,\ u+f\frac{t}{n},\ u+\frac{2ft}{n},\ \ldots\ldots\ldots\ldots\ u+\frac{(n-1)ft}{n}.$$

Hence, on the assumption that the velocity of the particle during each of these intervals was uniform and equal to that at the beginning of the corresponding interval, the total calculated distance that would be covered by the particle is

$$\begin{aligned}s_1 &= u\frac{t}{n}+\left(u+f\frac{t}{n}\right)\frac{t}{n}+\left(u+\frac{2ft}{n}\right)\frac{t}{n}+\cdots \text{ to } n \text{ terms}\\ &= u\frac{t}{n}\cdot n+\frac{ft^2}{n^2}(1+2+3+\cdots \text{ to } n-1 \text{ terms})\\ &= ut+\frac{ft^2}{n^2}\cdot\frac{n(n-1)}{2}\\ &= ut+\tfrac{1}{2}ft^2\left(1-\frac{1}{n}\right).\end{aligned}$$

Similarly the total distance that the particle would describe, as calculated on the assumption that during each of the above intervals the velocity of the particle was uniform, and equal to that at the end of the corresponding interval, is

$$\begin{aligned}s_2 &= \left(u+f\frac{t}{n}\right)\frac{t}{n}+\left(u+\frac{2ft}{n}\right)\frac{t}{n}+\cdots \text{ to } n \text{ terms}\\ &= \frac{ut}{n}\cdot n+\frac{ft^2}{n^2}\cdot(1+2+3+\cdots \text{ to } n \text{ terms})\\ &= ut+\frac{ft^2}{n^2}\cdot\frac{n(n+1)}{2}\\ &= ut+\tfrac{1}{2}ft^2\left(1+\frac{1}{n}\right).\end{aligned}$$

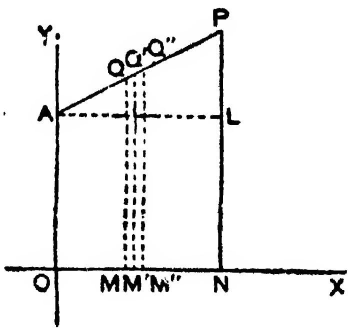
As actually during each interval the velocity gradually changes from its value at the beginning to that at the end, it is evident that the actual distance s is intermediate between s_1 and s_2, and this is true whatever value n may have. Making n infinitely large $\frac{1}{n}$ ultimately vanishes and s_1 and s_2 coincide, each being $ut+\frac{1}{2}ft^2$. Now s remaining always between s_1 and s_2, must coincide with this common value.

Hence, $s=ut+\frac{1}{2}ft^2$.

Proof by graphical method.

Let a particle move along a straight line with uniform acceleration f, and let u be its initial velocity, v its velocity after time t, and s the distance travelled over during this time.

Let two mutually perpendicular straight lines OX and OY be taken as axes of reference, and let time be measured along OX and the corresponding velocity of the moving particle along OY.



$OA=u$ along OY represents the velocity at zero-time. At any time t, represented by ON, the velocity v being represented by the corresponding ordinate PN, we have, $PN=v=u+ft$

$=OA+ft$.

$\therefore$ AL being drawn parallel to ON, $PL=ft$.

$\therefore$ $\tan PAL=\frac{PL}{AL}=\frac{ft}{t}=f$, a constant independent of t.

Hence, $\angle PAL$ is fixed for all positions of P.

i.e., the velocity-time graph of the moving point is here a straight line AP.

At any time represented by OM, the velocity is given by QM, and after a short interval MM', it changes to $Q'M'$. If MM' be infinitely

small, $Q'M'$ is very nearly equal to QM, and we can take both of them ultimately equal to $\frac{1}{2}(QM+Q'M')$. Hence, the distance described in time MM' is graphically represented ultimately by $\frac{1}{2}(QM+Q'M').MM'$ *i.e.*, by the area of the trapezium QM'. Similarly, the distance travelled during the next infinitesimal interval $M'M''$ is given graphically by the area of the trapezium $Q'M''$, and so on.

Thus the total distance travelled by the particle during the interval t $(=ON)$, is given graphically by the area of the trapezium $OAPN$, which

$$=\text{rectangle } OALN+\text{triangle } APL$$
$$=OA.ON+\tfrac{1}{2}AL.PL$$
$$=ut+\tfrac{1}{2}t.ft=ut+\tfrac{1}{2}ft^2.$$

Cor. The average velocity of a particle moving along a straight line with a uniform acceleration during any interval of time

= (*i*) *the velocity at the middle of the interval*

= (*ii*) *the mean of the initial and final velocities.*

For in this case,

$$\text{average velocity}=\frac{s}{t}=\frac{ut+\frac{1}{2}ft^2}{t}=u+\tfrac{1}{2}ft$$
$$=\frac{2u+ft}{2}=\frac{u+(u+ft)}{2}=\frac{u+v}{2}.$$

III. *To prove the formula* $v^2=u^2+2fs$.

Let a particle be moving along a straight line with a uniform acceleration f, and let u and v be its velocities at the beginning and end of any interval t during its motion, s being the distance passed over during this interval.

Then we know that

$$v=u+ft$$

and $$s=ut+\tfrac{1}{2}ft^2.$$

Hence, $$v^2=(u+ft)^2=u^2+2uft+f^2t^2$$
$$=u^2+2f(ut+\tfrac{1}{2}ft^2)$$
$$=u^2+2fs.$$

4·5. Space described in any particular second.

u being the initial velocity of a particle moving along a straight line with a uniform acceleration f, the space s_t described during the t^{th} second may be obtained by taking the difference of the total space described in t seconds and the total space described in $t-1$ seconds, both reckoned from the initial instant.

Thus,

$$s_t = (ut + \tfrac{1}{2}ft^2) - \{u(t-1) + \tfrac{1}{2}f(t-1)^2\}$$
$$= \mathbf{u + \tfrac{1}{2}f(2t-1)}.$$

4·6. Illustrative Examples.

Ex. 1. *A bullet passes through a wall* 9·6 *inches thick and its velocity changes from* 1200 *to* 800 *ft./sec. thereby. Find the time required by the bullet to pass through the wall, and the velocity when half the wall is penetrated.*

Let f be the retardation in ft.-sec. units to the motion of the bullet due to the resistance of the wall, supposed uniform.

Then from the given condition,

$$800^2 = 1200^2 - 2 \times f \times \left(\frac{9·6}{12}\right),$$

$$\text{or,} \quad 1·6f = 1200^2 - 800^2. \quad \cdots \quad \text{(i)}$$

Now t secs. being the time taken by the bullet to pass through the wall,

$$800 = 1200 - ft,$$

$$\text{or,} \quad ft = 1200 - 800. \quad \cdots \quad \text{(ii)}$$

Dividing (ii) by (i),

$$\frac{t}{1·6} = \frac{1}{1200+800}, \text{ or, } t = \frac{1·6}{2000} = ·0008 \text{ secs.}$$

Again, if v denotes the velocity when half the wall *i.e.*, 4·8 inches ($= ·4$ ft.) is penetrated

$$v^2 = 1200^2 - 2 \times f \times ·4$$
$$= 1200^2 - \tfrac{1}{2}(1200^2 - 800^2) \quad [\textit{from } (i)]$$
$$= 400^2\{9 - \tfrac{1}{2}(9-4)\} = 400^2 \times \tfrac{13}{2}.$$

$$\therefore \quad v = 400\sqrt{\frac{13}{2}} = 200\sqrt{26} = 1020 \text{ ft./sec. nearly.}$$

Ex. 2. *A particle moving along a straight line with uniform acceleration, describes 7 feet during the 5^{th} second of its motion and ultimately comes to rest after some time. If it describes $\frac{1}{64}$th of the whole distance during the last second of its motion, find how long it was in motion and also its initial velocity.*

Let u be the initial velocity and f the acceleration of the particle in ft.-sec. units, and t secs. the time for which it was in motion.

Then from the given conditions, we get,

$$7=u+\tfrac{1}{2}f(2.5-1)=u+\tfrac{9}{2}f \qquad \cdots \text{(i)}$$

$$0=u+ft \qquad \cdots \text{(ii)}$$

and $$u+\tfrac{1}{2}f(2t-1)=\tfrac{1}{64}(ut+\tfrac{1}{2}ft^2). \qquad \cdots \text{(iii)}$$

From (iii), using (ii), we get

$$-\tfrac{1}{2}f=\tfrac{1}{64}t(u+\tfrac{1}{2}ft)=\tfrac{1}{64}t(-\tfrac{1}{2}ft).$$

$$\therefore\ t^2=64. \quad \therefore\ t=8 \text{ secs.}$$

Now from (i) and (ii), subtracting,

$$7=\tfrac{9}{2}f-8f=-\frac{7f}{2}.$$

$$\therefore\ f=-2 \text{ ft /sec. units.}$$

Hence, $$u=-ft=16 \text{ ft./sec.}$$

Ex. 3. *A train travels from a station A to a station B from rest to rest. At a point C, somewhere between A and B it attains its highest speed of 60 miles per hour. If it travels with uniform acceleration from A to C, and with uniform retardation from C to B, find the distance between A and B, if the total journey takes 10 minutes.*

Let f_1 be the acceleration from A to C, f_2 the retardation from C to B. Also let s_1 and t_1 be the distance and time from A to C, and s_2 and t_2 those from C to B.

Using mile and hour as the units for distance and time we get, since velocities at A and C are zeros,

$$60=f_1t_1, \qquad 60^2=2f_1s_1,$$

and $$0=60-f_2t_2, \quad 0=60^2-2f_2s_2.$$

$$\therefore\ t_1=\frac{60}{f_1}, \qquad s_1=\frac{60^2}{2f_1}.$$

$$t_2=\frac{60}{f_2}, \qquad s_2=\frac{60^2}{2f_2}.$$

$$\therefore \quad 60\left(\frac{1}{f_1}+\frac{1}{f_2}\right)=t_1+t_2=\text{total time of journey}=\tfrac{1}{6},$$

and $$\frac{60^2}{2}\left(\frac{1}{f_1}+\frac{1}{f_2}\right)=s_1+s_2=s, \text{ the total distance from } A \text{ to } B.$$

$\therefore$ dividing, $6s=\frac{60}{2}=30,$

or, $s=5$ miles.

Ex. 4. *One motor-cycle M_1 stands 10 yds., in front of another M_2. Both start from rest; if M_1 moves off with uniform acceleration of 4 ft. per sec², and M_2 runs with a uniform velocity of 10 ft./sec., is it possible for M_2 to overtake M_1?*

What happens, if M_2 runs at a uniform rate of 16 ft./sec.?

10 yds. $=30$ ft. is the distance M_1 is ahead of M_2. In the first case after t secs. from start M_1 moves through a distance $\frac{1}{2}\times 4\times t^2=2t^2$ feet and M_2 moves over a length $10t$ feet. Hence the distance between them is $30+2t^2-10t$.

If M_2 is to overtake M_1, this should be zero.

Thus, $30+2t^2-10t=0$, or, $t^2-5t+15=0$.

The corresponding values of $t=\dfrac{5\pm\sqrt{5^2-4.15}}{2}$ are imaginary. Hence there is no real time when M_2 can overtake M_1; in other words, M_2 will never overtake M_1.

[*Alternatively,* distance between them, $30+2t^2-10t=\frac{35}{2}+2(t-\frac{5}{2})^2$ which cannot be zero for any real value of t.]

In the second case, the distance travelled by M_2 in t secs. being $16t$, the time when M_2 can overtake M_1 will be given by

$$30+2t^2-16t=0, \quad \text{or,} \quad t^2-8t+15=0$$

giving real values of t, namely 3 and 5 secs. The meaning of the double answer is that 3 secs. after start M_2 will overtake M_1 and leave him behind, but the velocity of M_1 continually increasing, after a further period of 2 secs. (*i.e.*, at 5 secs. from start), M_1 will again overtake M_2, and finally leave it behind, never to be overtaken by it any more.

Thus in this case M_2 meets M_1 twice during the motion.

Examples on Chapter IV(A)

1. A body has a velocity of 15 ft.-sec units at a certain instant, and 10 secs. later has a velocity of 45 ft.-sec. units. If the velocity changes uniformly, find the space described.

2. A tram car has its velocity uniformly increased from 10 ft. per sec. to 20 ft. per sec. while passing over 50 ft. Find the acceleration.

3. A train travelling 30 miles an hour is brought uniformly to rest at a station in $1\frac{1}{2}$ minutes. At what distance from the station were the brakes applied? What was the retardation in ft. per sec. per sec.?

4. A ball rolling down a slope with uniform acceleration passes three posts driven in the ground at equal distances. The velocities when passing the three successive posts are x, y, z. Prove that x^2, y^2, z^2 are in A.P.

5. A bullet fired into a target loses half its velocity after penetrating 3 inches. How much further will it penetrate? [*C. U. 1943*]

6. A particle starting with a given velocity moves for 3 secs. with constant acceleration, during which time it describes 81 ft.; the acceleration then ceases and during the next 3 secs., it describes 72 ft. Find its initial velocity and acceleration. [*U. P. 1939*]

7. A particle starts with an initial velocity u and passes successively over the two-halves of a given distance with accelerations f and f' respectively. Show that the final velocity is the same as if the whole distance were traversed with uniform acceleration $\frac{1}{2}(f+f')$. [*C. U. 1940*]

8. A cat, seeing a mouse at a distance of 15 ft. before it, starts from rest with an acceleration of 2 ft. per sec. per sec. and pursues it. If the mouse be moving uniformly with a velocity of 14 ft. per sec., find when and where the cat will catch the mouse.

9. A train is observed to take 50 secs. to pass from A to B, a distance of $\frac{1}{2}$ mile, and again to take the same time to

pass from B to C, a distance of $\frac{3}{4}$ mile. Find the velocities at A and C, in miles per hour, assuming that the accelera- tion of the train is uniform throughout.

10. The Bombay Mail starts from Howrah and stops at Burdwan The velocity increases uniformly till it reaches a maximum velocity V and then decreases uniformly. Show that the time taken by the train to run from Howrah to Burdwan is $\frac{2x}{V}$, where x is the distance between the two stations.

11. A train travels from a station X to a station Y in 45 minutes. At a point Z, somewhere between X and Y, it attains its maximum velocity of 45 miles per hour. If it travels with uniform acceleration from X to Z and uniform retardation from Z to Y, find the distance between X and Y, it being supposed that the train starts from rest at X and comes to rest at Y.

12. A train running with uniform acceleration passes by two stations A and B with velocities u and v. Is the velo- city of the train at half-time equal to, greater than or less than the velocity half-way ?

13. A point is moving with uniform acceleration ; in the eleventh and fifteenth seconds from the commencement of the motion it moves through 720 and 960 centimetres res- pectively. Find the distance covered by it in 20 seconds.

14. If a, b, c be the spaces described in the pth, qth and rth seconds by a body starting with a given velocity u and moving with uniform acceletation f, show that

$$a(q-r)+b(r-p)+c(p-q)=0.$$

[C. P. 1962 ; C. H. 1967]

15. A train stopping at two stations 2 miles apart takes 4 minutes on the journey from one of the stations to the other. Assuming that its motion is first that of uniform acceleration x and then that of uniform retardation y, prove that $\frac{1}{x}+\frac{1}{y}=4.$

a mile and a minute being the unit of distance and time respectively. [C. U. 1934]

16. In a racing competition two cars start off together from the same point with velocities u_1 and u_2 and move along parallel lines with acceleration f_1 and f_2 respectively. If they reach their destination at the same time, show that the distance traversed is $\frac{2(u_1-u_2)(u_1f_2-u_2f_1)}{(f_1-f_2)^2}$.

17. A train travels from rest at one station to rest at another (in the same straight line) distant d ft. It moves for first part of the distance with an acceleration of a ft. per sec^2 and for the remainder with a retardation of b ft. per sec^2. Show that it will accomplish the journey in

$$\sqrt{\left\{\frac{2(a+b)d}{ab}\right\}} \text{ secs.}$$ [C. U. 1945]

18. If v_1, v_2, v_3 be the average velocities in three successive intervals of time t_1, t_2, t_3 of a point moving in a straight line with uniform acceleration, show that

$$\frac{v_1-v_2}{v_2-v_3}=\frac{t_1+t_2}{t_2+t_3}.$$

19. A bicyclist running with a uniform velocity of 20 ft. per sec. is 84 ft. behind an engine, which is just starting from rest with a uniform acceleration of 2 ft. sec. units. When will the cyclist meet the engine ? Explain the double answer.

20. An express train is overtaking a goods train on the same line, their velocities being u_1 and u_2 respectively. When there is a distance x between them, each is seen from the other. Prove that it is just possible to avoid a collision if $(u_1-u_2)^2=2(f_1+f_2)x$, when f_1 is the greatest retardation and f_2 the greatest acceleration which can be produced in the two trains respectively.

21. Two trains on the same line are approaching one another with velocities u_1 and u_2 respectively. When there is a distance x between them each is seen from the other. Prove that it is just possible to avoid a collision if

$$u_1^2f_2+u_2^2f_1=2f_1f_2x,$$

where f_1 and f_2 are the greatest retardations which the brakes can produce in the respective trains.

22. Two particles P and Q, start together and move from the same point A along the same line AB. P has a uniform velocity of 20 ft. per sec., and Q a uniform acceleration of 4 ft. per sec^2 and no initial velocity. Find when and where they meet again. Before they meet again, find when the distance between the two will be maximum and what is the maximum distance.

23. A particle is projected in a straight line with a certain velocity and a constant acceleration. One second later, another particle is projected after it with half the velocity and double the acceleration. When the second particle overtakes the first, their velocities are 31 and 22 ft. secs. respectively. Prove that the distance traversed is 48 ft.

24. If a point moving under uniform acceleration describes successive equal distances in times t_1, t_2, t_3, then

$$\frac{1}{t_1} - \frac{1}{t_2} + \frac{1}{t_3} = \frac{3}{t_1 + t_2 + t_3}.$$

25. A train starting from Sealdah stops at Ranaghat. It moves with uniform acceleration for the first quarter of the journey, with uniform retardation for the last quarter, and with uniform velocity during the middle half of the journey. Show that the average velocity of the train is $\frac{2}{3}$ of its maximum velocity.

26. A bus starts from rest with an acceleration of 1 ft. per sec^2. Show that a passenger who can run at the rate of 9 ft. per sec. cannot catch the bus if he is more than $40\frac{1}{2}$ ft. behind it.

27. A train travels in 6 minutes a distance of 2 miles between two stations, starting at rest and finishing at rest. If it moves with uniform acceleration for the first two-thirds of the journey and with uniform retardation for the remainder, find the acceleration, the retardation and the maximum velocity.

28. A distance s is divided into n equal parts at the end of each of which the acceleration of a moving particle is

increased by f/n; show that the velocity of the particle after describing the distance is

$$\sqrt{fs\left(3-\frac{1}{n}\right)},$$

where f is the initial acceleration of the particle starting from rest.

29. The velocity of a train increases at a constant rate f_1 from 0 to v, then remains constant for an interval, and finally decreases to 0 at the constant rate f_2. If x be the total distance described, prove that the total time taken is

$$\frac{x}{v}+\frac{v}{2}\left(\frac{1}{f_1}+\frac{1}{f_2}\right).$$

30. A body moves in a straight line AB and its distance from A after t secs. is s ft. If s and t satisfy the relation

$s = 0.25t + 0.375t^2$, for values of t,

prove from definition only (without assuming any formula) that the acceleration is uniform.

Find also, (i) the velocity at the end of 4 secs.

(ii) the average velocity during the 4th sec.

31. A constable seeing a thief at a distance x ft. starts with velocity u and moves with acceleration α in order to catch him, whilst the thief runs with acceleration β, starting from rest. Show that the constable will overtake the thief either if $\alpha \geqslant \beta$ or, if $\alpha < \beta < \alpha + \dfrac{u^2}{2x}$.

32. Two particles move in the same straight line with constant accelerations f and f'. If their velocities be u and u' at a certain instant when they are at distances a and a' from some fixed point on the line, prove that they cannot pass each other more than twice; and if they do so twice, the interval between the two times of passing is

$$\frac{2}{f-f'}\sqrt{\{(u-u')^2-2(a-a')(f-f')\}}.\quad [\text{C. U. 1938}]$$

33. Prove that for a particle moving with uniform acceleration f in a straight line,

$$f = 2\left(\frac{s'}{t'} - \frac{s}{t}\right)\Big/(t+t'),$$

where s is the space described in t seconds, and s' during the next t' seconds.

34. A point moves in a straight line with a constant acceleration. If the distances of the moving point from a fixed point on the line be x_1, x_2, x_3 at the instants t_1, t_2, t_3, prove that the acceleration is

$$2\left\{\frac{(x_2 - x_3)t_1 + (x_3 - x_1)t_2 + (x_1 - x_2)t_3}{(t_2 - t_3)(t_3 - t_1)(t_1 - t_2)}\right\}$$

ANSWERS

1. 300 ft. 2. 3 ft/sec^2. 3. $\frac{3}{8}$ mile ; $\frac{11}{45}$ ft/sec^2.
5. 1 inch. 6. 30 ft/sec. ; -2 ft/sec^2.
8. After 15 secs. at a distance 225 ft. from its starting point.
9. 27 miles/hr. ; 63 miles/hr. 11. $16\frac{7}{8}$ miles.
12. Less. 13. 13800 cms.
19. 6 secs. ; 14 secs.
22. After 10 secs. from start, at a distance 200 ft. ; 5 secs. from start ; 50 ft. 27. $\frac{11}{45}$ ft/sec^2 ; $\frac{22}{45}$ ft/sec^2 ; 40 miles/hr.
30. (i) 3·25 ft/sec. ; (ii) 2·875 ft/sec.

4·7. Analytical treatment of motion in a straight line.

A. Analytical expressions for velocity and acceleration.

Let the distance of a moving point P from a fixed point O be x at any time t, and let Q be its position at time $t + \Delta t$,

where $OQ = x + \Delta x$, so that $PQ = \Delta x$. Since velocity is the rate of displacement, the velocity v of P at time t

$$= \underset{\Delta t \to 0}{Lt} \frac{PQ}{\Delta t} \text{ i.e., } = \underset{\Delta t \to 0}{Lt} \frac{\Delta x}{\Delta t}$$

$$= \frac{dx}{dt} \text{ (in the notation of Calculus).}$$

Thus, $$\mathbf{v} = \frac{\mathbf{dx}}{\mathbf{dt}}. \qquad \cdots \quad (1)$$

Let v be the velocity of the particle at time t and $v + \Delta v$ be its velocity at time $t + \Delta t$.

Since acceleration of a particle is the rate of change of velocity, therefore,

acceleration f at time t

$$= \underset{\Delta t \to 0}{Lt} \frac{(v + \Delta v) - v}{\Delta t} = \underset{\Delta t \to 0}{Lt} \frac{\Delta v}{\Delta t}$$

$$= \frac{dv}{dt} \text{ (in the notation of Calculus).}$$

Since $$v = \frac{dx}{dt}, \qquad \text{hence, } f = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2}.$$

Also, $$f = \frac{dv}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt} = v\frac{dv}{dx}.$$

Thus, the acceleration of a particle may be expressed as

$$\mathbf{f} = \frac{\mathbf{dv}}{\mathbf{dt}} = \frac{\mathbf{d^2x}}{\mathbf{dt^2}} = \mathbf{v}\frac{\mathbf{dv}}{\mathbf{dx}}. \qquad \cdots \quad (2)$$

Note 1. Of the above three forms of f, as to which one is the best to employ in a given case, will become clear from the illustrative examples given below.

Note 2. $\frac{dv}{dt}, \frac{dx}{dt}, \frac{d^2x}{dt^2}$ etc. are very often denoted by $\dot{v}, \dot{x}, \ddot{x}$ etc.

B. Motion with constant acceleration.

A particle moves along a straight line from a fixed point on it with initial velocity u and constant acceleration f. To find the velocity v and the distance s travelled after any time t.

Here, the equation of motion is

$$\frac{dv}{dt}=f. \qquad \cdots \quad (1)$$

Integrating with respect to t, we have

$$v=ft+A \qquad \cdots \quad (2)$$

where A is an arbitrary constant of integration.

It is given that initially, *i.e.*, when $t=0$, velocity $v=u$.

$\therefore$ from (2), $u=f.0+A$.

$\therefore$ from initial condition, we get $A=u$.

$\therefore$ (2) becomes $\mathbf{v=u+ft}$. $\qquad \cdots \quad (3)$

Again, we have the equation of motion,

$$\frac{d^2s}{dt^2}=f. \qquad \cdots \quad (4)$$

Integrating (4) with respect to t, we get

$$\frac{ds}{dt}=ft+B. \qquad \cdots \quad (5)$$

From the given condition, initially, *i.e.*, when $t=0$, $\frac{ds}{dt}$ *i.e.*, velocity $=u$.

$\therefore$ from (5), $u=f.0+B$. $\therefore$ $B=u$.

$\therefore$ (5) becomes $\frac{ds}{dt}=u+ft$. $\qquad \cdots \quad (6)$

Integrating (6) with respect to t, we get

$$s=ut+\tfrac{1}{2}ft^2+C. \qquad \cdots \quad (7)$$

From the given condition, when $t=0$, $s=0$, hence, $C=0$.

$\therefore$ (7) becomes

$$\mathbf{s=ut+\tfrac{1}{2}ft^2}. \quad \cdots \quad (8)$$

Again, the equation of motion can be written as

$$v\frac{dv}{ds}=f \quad \cdots \quad (9)$$

$$\text{or,} \quad \tfrac{1}{2}\frac{d}{ds}(v^2)=f. \quad \cdots \quad (10)$$

Integrating (10) with respect to s, we have

$$\tfrac{1}{2}v^2=fs+C'. \quad \cdots \quad (11)$$

From the given condition, initially, *i.e.*, when $s=0$, $v=u$.

$\therefore$ (11) gives $\tfrac{1}{2}u^2=0+C'$. $\therefore$ $C'=\tfrac{1}{2}u^2$.

$\therefore$ (11) becomes $\tfrac{1}{2}v^2=fs+\tfrac{1}{2}u^2$,

$$\text{or,} \quad \mathbf{v^2=u^2+2fs}. \quad \cdots \quad (12)$$

Cor. If the particle starts from rest so that $u=0$, the equations (3), (8), (12) take the simpler forms

$$v=ft,\ s=\tfrac{1}{2}ft^2,\ v^2=2fs.$$

4·8. Illustrative Examples.

Ex. 1. *A particle is moving in a straight line, and its distance in feet from a given point in the line after t secs. is given by*

$$x=6+3t+2t^3.$$

Find the velocity at the end of 2 secs. and acceleration at the end of 4 secs.

Differentiating $x=6+3t+2t^3$ with respect to t,

$$\frac{dx}{dt}=3+6t^2. \quad \cdots \quad (1)$$

Putting $t=2$ in (1), $\frac{dx}{dt}$ *i.e.*, velocity at the end of 2 seconds

$$=3+6.2^2=27 \text{ ft/sec}.$$

Differentiating (1) with respect to t again,

$$\frac{d^2x}{dt^2}=12t. \qquad \cdots \quad (2)$$

Putting $t=4$ in (2), $\frac{d^2x}{dt^2}$ *i.e.*, the acceleration at the end of 4 secs.

$$=12.4=48 \text{ ft/sec}^2.$$

Ex. 2. *A particle moves in a straight line according to the law*

$$v^2=6a\,(x \sin x+\cos x),$$

where x is its distance from a fixed point on the line; find its acceleration.

Here, $\quad v^2=6a(x \sin x+\cos x). \qquad \cdots \quad (1)$

Differentiating (1) with respect to x, we get

$$2v\frac{dv}{dx}=6a\,(x \cos x+\sin x-\sin x).$$

$$\therefore \quad \text{acceleration}=f=v\frac{dv}{dx}=3ax \cos x.$$

Ex. 3. *The law of motion of a body moving along a straight line is $x=\frac{1}{2}vt$; prove that its acceleration is constant.*

Given $\quad x=\frac{1}{2}vt, \qquad \cdots \quad (1)$

differentiaing with respect to t,

$$\frac{dx}{dt}=\tfrac{1}{2}v+\tfrac{1}{2}\frac{dv}{dt}t,$$

$$\text{or,} \qquad v=\tfrac{1}{2}v+\tfrac{1}{2}\frac{dv}{dt}t, \quad \textit{i.e.,} \quad v=\frac{dv}{dt}t. \qquad \cdots \quad (2)$$

Differentiating respect to t again,

$$\frac{dv}{dt}=\frac{d^2v}{dt^2}\cdot t+\frac{dv}{dt}.$$

$$\therefore \quad \frac{d^2v}{dt^2}t=0, \quad \textit{i.e.,} \quad \frac{d^2v}{dt^2}=0. \quad (\text{as } t\neq 0).$$

On integration, $\frac{dv}{dt}=$ constant.

$\therefore$ the acceleration is constant.

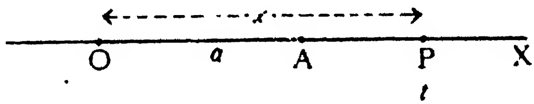
Ex. 4. *Prove that it is impossible for a particle to move in a straight line so that its velocity varies as the distance described from the commencement of the motion.*

If possible, let $v = \lambda x$; $\quad\cdots\quad$ (1)

$$\text{hence, } f = \text{accl.} = \frac{dv}{dt} = \lambda\frac{dx}{dt} = \lambda v = \lambda^2 x \ (\textit{from } 1).$$

Hence, initially when $x = 0$, both v and f are zero ; so that the particle remains at rest.

Ex. 5. *A particle starts from rest at a distance a from a fixed point O, and moves with an acceleration proportional to its distance from O, away from it. Investigate its motion.* [*C. H. 1958*]



Let x be the distance of the particle from O at any time t. Its acceleration away from O is then μx (say, $\mu > 0$).

The equation of motion is given by

$$\frac{d^2x}{dt^2} = \mu x. \qquad \cdots \qquad \cdots \quad \text{(i)}$$

Multiplying both sides by $2\dfrac{dx}{dt}$ and integrating with respect to t, we get

$$\left(\frac{dx}{dt}\right)^2 = \mu x^2 + C, \qquad \cdots \qquad \cdots \quad \text{(ii)}$$

where C is an integration constant.

Since initially, when $x = a$, the velocity *i.e.*, $\dfrac{dx}{dt} = 0$, from (ii)

$$0 = \mu a^2 + C, \text{ or } C = -\mu a^2.$$

$\therefore$ from (ii), using this value of C, we get velocity at any instant

$$\left(\frac{dx}{dt}\right)^2, \ i.e., \ v^2 = \mu(x^2 - a^2). \quad \cdots \qquad \cdots \quad \text{(iii)}$$

$\therefore \qquad \dfrac{dx}{dt} = \sqrt{\mu}\sqrt{x^2 - a^2}$

or, $\qquad \dfrac{dx}{\sqrt{x^2 - a^2}} = \sqrt{\mu}\, dt.$

$\therefore$ integrating (putting $x = a \cosh \theta$ on the left side)

$$\cosh^{-1}\frac{x}{a} = \sqrt{\mu}t + D, \qquad \cdots \qquad \cdots \quad \text{(iv)}$$

where D is an integration constant.

Since $\quad x = a$ when $t = 0$, from (iv), $0 = D$.

$\therefore$ (iv) gives at any instant

$$x = a \cosh \sqrt{\mu}t. \qquad \cdots \qquad \cdots \quad \text{(v)}$$

Differentiating with respect to t,

$$\frac{dx}{dt} \text{ i.e., } v = a\sqrt{\mu} \sinh \sqrt{\mu}t. \qquad \cdots \quad \text{(vi)}$$

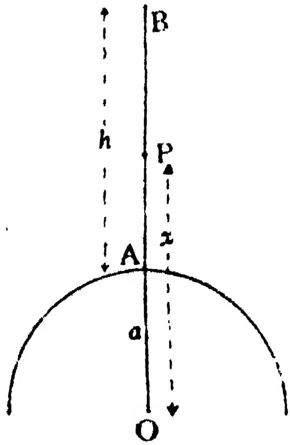
The equation (iii) gives velocity at any distance x, (v) gives position at time t, and (vi) gives velocity after any time t.

Note. The *general solution* of (i) [writing $-\mu A^2$ for C in (ii), where A is arbitrary] is $\mathbf{x = A \cosh (\sqrt{\mu}t + D)}$. For any given initial conditions, from the given values of x and of the velocity v, *i.e.*, $\dfrac{dx}{dt}$ at time $t = 0$, the integration constants A and D will be definitely determined, and the motion completely known.

Putting $\frac{1}{2}Ae^{D} = A_1$ and $\frac{1}{2}Ae^{-D} = B_1$, *another form of the general solution* of (i) is $\mathbf{x = A_1 e^{\sqrt{\mu}t} + B_1 e^{-\sqrt{\mu}t}}$, where A_1 and B_1 are arbitrary constants.

Ex. 6. *The acceleration due to attraction of the earth (assumed to be a sphere of radius a) on a particle outside it varies inversely as the square of its distance from the centre, and the value of the acceleration due to gravity at the surface of the earth is g. A particle is dropped*

from a height h above the earth's surface. Find its velocity on reaching the earth's surface.



Let O, the centre of the earth, be chosen as origin, and let the particle dropped from B at a height h above the earth's surface (*i.e.*, at a distance $a+h$ from O), falling towards O under earth's attraction, be at P at a distance x from O after time t.

Let the acceleration of the particle at this point due to earth's attraction be $\frac{\mu}{x^2}$. At the surface where $x=a$, the value being given to be g, $\frac{\mu}{a^2}=g$, or, $\mu=ga^2$. Thus the acceleration at P is $\frac{ga^2}{x^2}$ towards O, *i.e.*, $-\frac{ga^2}{x^2}$ along OP. The mathematical expression for acceleration along OP (*i.e.*, in the direction of x increasing) being known to be $\frac{d^2x}{dt^2}$, we get here the equation of motion to be

$$\frac{d^2x}{dt^2}=-\frac{ga^2}{x^2}. \qquad \cdots \qquad \cdots \quad (1)$$

Multiplying both sides by $2\frac{dx}{dt}$, and integrating with respect to t,

$$\left(\frac{dx}{dt}\right)^2=\frac{2ga^2}{x}+A, \qquad \cdots \qquad \cdots \quad (2)$$

where A is the constant of integration.

Initially, when $x=a+h$, $\frac{dx}{dt}$ (which represents the velocity) is 0, the particle being dropped from rest.

$$\therefore \qquad 0=\frac{2ga^2}{a+h}+A, \text{ or } A=-\frac{2ga^2}{a+h}.$$

Hence, from (2), at any point

$$\left(\frac{dx}{dt}\right)^2=2ga^2\left(\frac{1}{x}-\frac{1}{a+h}\right). \qquad \cdots \quad (3)$$

When the particle reaches the surface of the earth, *i.e.*, when $x=a$, the velocity v is given by

$$v^2=2ga^2\left(\frac{1}{a}-\frac{1}{a+h}\right)=\frac{2gha}{a+h}.$$

Examples on Chapter IV(b)

1. A particle is moving in a straight line, and its distance in feet from a given point in the line after t seconds from start is given by $x=t^3-2t-16$.

Find its velocity at the end of 3 seconds, and acceleration at the end of 4 seconds.

What is its acceleration when it is at a distance 5 ft. from the given point ?

2. A particle moves along a straight line, and its distance from a fixed point on the line after t seconds from start is given by $x=a+bt+ct^2$. Prove that it moves with a constant acceleration.

3. The law of motion of a body moving along a straight line is $x=\frac{1}{2}vt$, x being its distance from a fixed point on the line at time t, and v its velocity there ; prove that it moves with a constant acceleration.

4. A straight rod AB slides on a plane with its ends on two fixed perpendicular lines OX, OY. Show that the speeds of the extremities of the rod are inversely proportional to their distances from O.

5. Two men, A and B, are walking along two roads which meet at right angles at C, A approaching and B receding from C ; prove that if they are always the same distance apart, A's velocity must be to B's velocity at each instant as $CB : CA$ at that instant.

6. A particle moves along a straight line, and at a distance x from a fixed point O on the line its velocity is $\mu\sqrt{\frac{c-x}{x}}$. Prove that its acceleration is directed towards O and is inversely proportional to the square of the distance.

7. A particle moves from rest at a distance c from a fixed point O, with an acceleration $\frac{\mu}{x^2}$ away from O at a distance x. Find its velocity when it is at a distance $2c$ from O.

8. A particle of mass m moves in a straight line under an acceleration n^2x towards a fixed point on the line when at a distance x from it. If it be initially projected with a velocity V towards the point from a distance a from it, prove that it reaches the point after a time $\frac{1}{n}\tan^{-1}\frac{na}{V}$.

9. A particle moves in a straight line with an acceleration n^2x away from a fixed point O on the line, x being the distance of the particle from O and n being a constant.

If $x=a$ and $\frac{dx}{dt}=V$ at time $t=0$, show that at time t,

$$x=a\cosh nt+\frac{V}{n}\sinh nt.$$

10. A particle, moving along the x-axis, has an acceleration towards the origin which is given by the formula $\frac{4n^2}{x^2}$ for $x \geqslant 2$, and by the formula $\frac{n^2x}{2}$ for $x < 2$.

If the particle starts from rest at the point $x=4$, prove that it will reach the origin with velocity $2n$.

ANSWERS

1. 25 ft./sec.; 24 ft./sec^2; 18 ft./sec^2. 7. $\sqrt{\frac{\mu}{c}}$.

CHAPTER V

RECTILINEAR MOTION UNDER GRAVITY

51. Acceleration due to gravity.

If a heavy body is dropped from any height, it falls vertically towards the earth, and it may be noticed that its velocity, which is initially zero, continually increases as it falls, or in other words, it falls with an acceleration. This phenomenon is attributed to the attraction of the earth on the body, which goes by the name of earth's gravitation.

Now, if observation be made by dropping it from different heights,* and the corresponding times of reaching the ground be noted by a stop-watch, it will be found that the distance through which the body falls from rest is proportional to the square of the time of falling, in other words, $s = kt^2$. But this is only possible when the acceleration of the falling body is uniform, (which can be proved even from fundamental considerations, using definitions only, without assuming any formula). We thus conclude that *when a body is dropped, it falls vertically towards the earth with a constant acceleration.*

If again, two different bodies, say a heavy piece of stone and a light bit of paper, be dropped from the same height, it

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* Towards the close of the 16th century, Galileo for the first time performed this experiment in a modified form. In order to avoid the difficulty of observing the time which is extremely short in the case of a freely falling body, he allowed a sphere to roll down along an inclined plane, and noted the times of describing different distances marked along it, when he found results similar to the above case to hold, and concluded that acceleration of the body down the plane was uniform. Repeating the same experiment with different inclinations of the plane to the horizon, he finally deduced the conclusion in the case of vertical falling.

is usually observed that the heavier body reaches the ground quicker than the lighter. This difference however is due to the resistance of the surrounding air. The well-known *Guinea and feather experiment of Newton* (in which a guinea and a feather were observed by dropping them simultaneously from the same point inside a long glass tube from which air had been pumped out previously by an air-pump) clearly demonstrated that in the absence of air resistance, different bodies dropped from the same height reach the ground simultaneously and as each moves with a constant acceleration as mentioned before, it follows that at the same place on earth *this acceleration is the same for all falling bodies.*

When a body is projected upwards, it is observed that its velocity gradually diminishes, or in other words, it possesses an acceleration in the opposite direction, *i.e.*, vertically downwards, the magnitude of which is the same as that of a falling body.

The above experiments, as also more careful and accurate experiments of modern times lead finally to the following conclusions :

A body free to move under the influence of earth's attraction, whether rising or falling, possesses a uniform acceleration which is vertically downwards, and this acceleration is the same for all bodies at the same place on the suaface of the earth. This vertically downwards acceleration is defined as the **acceleration due to gravity**, and is always denoted by '**g**'. Its value has been determined accurately by various methods, among which mention may be made of the well-known pendulum experiments. It is found to vary slightly from place to place on the surface of the earth, from 32·091 ft/sec^2 at the equator to 32·252 ft/sec^2 at the poles.* For numerical examples, this value in round figure is taken as **32 ft/sec^2** or **981 cms/sec^2**.

*This is due to the attraction of the earth being different at different distances from its centre, and the earth being not exactly round, but slightly flattened at the poles.

5·2. A body moving vertically downwards.

Taking the downward direction as positive, any problem in this case can be worked out with the help of the standard formulæ for uniformly accelerated motion in a straight line, only replacing f in the formulæ by g in this case.

For instance, if a body be dropped from a height h above the ground, the time taken to reach the ground is given by

$$h=\tfrac{1}{2}gt^2, \text{ or } t=\sqrt{\frac{2h}{g}}$$

and velocity on reaching the ground is given by

$$v^2=2gh, \quad \text{or } v=\sqrt{2gh}$$

which will be referred to as the velocity due to a fall through a height h.

5·3. A body projected vertically upwards.

In this case, taking the upward direction as positive, in the formulæ for uniformly accelerated motion in a straight line, f is to be replaced by $-g$.

Let u be the velocity with which a body is projected vertically upwards. As it rises, its velocity gradually diminishes, until it becomes zero, when the body is at its greatest height. After this the body begins to fall with a gradually increasing velocity.

(*i*) *Greatest height and the time of rise* :

Let H be the greatest height attained by the particle, and T the time to the greatest height.

Then, $\quad 0=u-gT$

and $\quad 0=u^2-2gH.$

$$\therefore \quad T=\frac{u}{g}, \quad H=\frac{u^2}{2g}.$$

(*ii*) *Time of fall, and the velocity on reaching the ground again* :

When the particle begins to fall, it is at a height $\frac{u^2}{2g}$ above the ground, and its velocity is zero.

Taking the downward direction as positive, now the acceleration is $+g$. If T' be the time of fall, and v be the velocity on reaching the ground again, we get

$$\frac{u^2}{2g} = \tfrac{1}{2}gT'^2$$

and $$v^2 = 2.g.\frac{u^2}{2g}.$$

Thus, $$v^2 = u^2, \quad \text{or,} \quad v = u,$$

and $$T'^2 = \frac{u^2}{g^2}, \quad \text{or,} \quad T' = \frac{u}{g} = T.$$

Thus for a body projected vertically upwards,

the time of rise = the time of fall,

the velocity on reaching the ground again

= the initial velocity of projection.

Note. Accordingly, *total time of flight* $= \frac{2u}{g}$.

(*iii*) *Time to a given height* :

Let a body be projected vertically upwards with a velocity u, and let t be the time at which it is at a given height h from the starting point. Taking upward direction as positive, the acceleration of the body is $-g$.

Hence, $$h = ut - \tfrac{1}{2}gt^2, \quad \text{or,} \quad \tfrac{1}{2}gt^2 - ut + h = 0.$$

$$\therefore \quad t = \frac{u \pm \sqrt{u^2 - 2gh}}{g} = \frac{u}{g} \pm \frac{\sqrt{u^2 - 2gh}}{g}.$$

Note. The reason for this double answer is that the body is at the same height h twice, once on its way up, the time for which is less than u/g, and once on its way down, for which the time is greater than u/g. The difference of the two times above from u/g being the same, it is once more demonstrated that *the time from any point on the path to the greatest height, and the time from the greatest height back to the same point are equal.*

Again, if $h > \frac{u^2}{2g}$, the above values of t are imaginary, showing that the particle does not attain any height greater than $\frac{u^2}{2g}$.

(*iv*) *Velocity at any height* :

In case of a body projected vertically upwards with a velocity u, if v be its velocity at a height h above the starting point, we get, taking the upward direction as positive,

$$v^2 = u^2 - 2gh,$$

or, $$v = \pm \sqrt{u^2 - 2hg}.$$

Note. The positive sign gives the velocity on its way up through the point, and the negative sign that on its way down, showing that *at the same point of its path, the magnitude of the velocity is the same when falling as when rising.* A particular case of this is that when reaching the starting point again the velocity is the same as the velocity of projection, as has been proved before.

5·4. Analytical Treatment.

A particle is projected vertically upwards with a given velocity ; to investigate its motion.

Let u be the velocity of projection.

Taking the upward direction as positive, the equation of motion is

$$\frac{d^2x}{dt^2} = -g. \qquad \cdots \quad (1)$$

Integrating with respect to t,

$$\frac{dx}{dt} = -gt + C \qquad \cdots \quad (2)$$

when $t = 0$, $\dfrac{dx}{dt} = u$, $\therefore$ $C = u$.

$$\therefore \quad \frac{dx}{dt} = u - gt. \qquad \cdots \quad (3)$$

$\therefore$ *if v be the velocity at time t,*

$$v = u - gt. \qquad \cdots \quad (4)$$

Now integrating (3) with respect to t,

$$x = ut - \tfrac{1}{2}gt^2 + D. \qquad \cdots \quad (5)$$

Initially *i.e.*, when $t=0$, $x=0$, $\therefore$ $D=0$,

$$\therefore \quad x = ut - \tfrac{1}{2}gt^2. \qquad \cdots \quad (6)$$

Hence, *if h be the height at time t,*

$$h = ut - \tfrac{1}{2}gt^2. \qquad \cdots \quad (7)$$

Again, multiplying both sides of (1) by $2\dfrac{dx}{dt}$, we have

$$\frac{d}{dt}\left\{\frac{dx}{dt}\right\}^2 = -2g\frac{dx}{dt}.$$

Now integrating with respect to t,

$$\left(\frac{dx}{dt}\right)^2 = -2gx + C.$$

Since initially *i.e.*, when $x=0$, $\dfrac{dx}{dt}=u$, $\therefore$ $C=u^2$.

$$\therefore \quad v^2 = u^2 - 2gx. \qquad \cdots \quad (8)$$

Hence, *if v be the velocity at height h,*

$$v^2 = u^2 - 2gh. \qquad \cdots \quad (9)$$

Now, we know that x will be maximum when $\dfrac{dx}{dt}=0$, $\dfrac{d^2x}{dt^2}$ is negative. From $\dfrac{dx}{dt}=0$, we get $0=u-gt$, *i.e.*, $t=\dfrac{u}{g}$, and then $x = u\cdot\dfrac{u}{g} - \dfrac{1}{2}g\cdot\dfrac{u^2}{g^2} = \dfrac{u^2}{2g}$.

Thus, if T be the *time to the greatest height* and H be the *greatest height*, then

$$\mathbf{T} = \frac{\mathbf{u}}{\mathbf{g}}$$

$$\mathbf{H} = \frac{\mathbf{u}^2}{\mathbf{2g}}.$$

Since velocity at time t is given by $u-gt$, hence when $t=\frac{u}{g}$, the velocity $=u-gt=0$.

Thus, *the velocity at the greatest height is zero.*

From (4) and (9), it is clear that as t and h gradually increase, v gradually diminishes and hence there comes a time and a height where this vertical upward velocity is just zero. This height is called the *greatest height* and this time is called *the time of rise.* If we denote by T the time of rise, and by H the greatest height, then putting $v=0$ in (4) and (9), we get

$$T=\frac{u}{g} \text{ and } H=\frac{u^2}{2g}. \qquad \cdots \quad (10)$$

After the particle reaches the greatest height, its upward vertical velocity being just zero, it is momentarily at rest and then falls freely vertically downwards on account of the force of gravity, reaches the starting point and may even go past that point.

From (7), we have $\frac{1}{2}gt^2-ut+h=0$.

$$\therefore \quad t=\frac{u\pm\sqrt{u^2-2gh}}{g}=\frac{u}{g}\pm\frac{\sqrt{u^2-2gh}}{g}. \qquad \cdots \quad (11)$$

From above, it is clear that for a given value of h, there are two values of t *i.e.*, the particle is at a given height twice, once on its way up, the time for which is less than u/g and next on its way down, for which the time is greater than u/g.

From (11), it follows when $h=0$, $t=0$, or $t=\frac{2u}{g}$.

This means that the particle is at the starting point twice, once initially and next when it comes back to it after attaining the greatest height and the second time is called the total time of flight.

$$\therefore \quad \textbf{Total time of flight}=\frac{2u}{g}. \qquad \cdots \quad (12)$$

Since time of rise $=\frac{u}{g}$ and total time of flight $=\frac{2u}{g}$,

hence it follows that time of fall $=\frac{u}{g}$.

$\therefore$ *time of rise = time of fall* $=\frac{u}{g}$. ... (13)

From (9), it follows that

$$v=\pm\sqrt{u^2-2gh}.$$

The positive sign gives the velocity on its way up and the negative sign on its way down, showing that *at the same point of its path, the magnitude of the velocity is the same when falling as when rising.*

When $h=0$, $v=\pm u$.

This means that

the velocity on reaching the ground again

= the initial velocity of projection.

Note. Taking the *downward direction as positive*, the above equation of motion (1) can be written as

$$\frac{d^2x}{dt^2}=g. \quad \cdots \quad (1)$$

During the downward motion, it is usually convenient to take the highest point as origin.

Illustration. Suppose we are required to find time which a particle takes when it is dropped freely from a height h above the ground.

Integrating (1) twice with respect to t, we have

$$\frac{dx}{dt}=gt+C \quad \cdots \quad (2)$$

$$x=\tfrac{1}{2}gt^2+Ct+D. \quad \cdots \quad (3)$$

Since initially, $\frac{dx}{dt}=0$, $\therefore$ $C=0$ and since $x=0$ when $t=0$,

$\therefore$ $D=0$.

$\therefore$ from (3), we get $x=\frac{1}{2}gt^2$.

$\therefore$ required time $=\sqrt{\frac{2h}{g}}$.

5·5. Illustrative Examples.

Ex. 1. *From a balloon ascending with a velocity of* 32 *ft/sec., a stone is let fall and reaches the ground in* 17 *secs. How high was the balloon when the stone was dropped ?*

At the instant when the stone was dropped, it was moving with the velocity of the balloon, namely 32 ft./sec. upwards, *i.e.*, -32 downwards, and its acceleration, when free, was $g=32$ ft./sec^2 downwards. Hence, h being the height of the balloon at the instant in question, this distances is described by the stone in 17 secs.

$$\therefore \quad h=-32\times 17+\tfrac{1}{2}\times 32\times 17^2$$
$$=4080 \text{ ft.}$$

Alternative method :

Taking the downward direction as positive and measuring x from the highest point, the equation of motion is

$$\frac{d^2x}{dt^2}=g \qquad \cdots \quad (1)$$

$$\therefore \quad \frac{dx}{dt}=gt+C \qquad \cdots \quad (2)$$

when $\quad t=0, \quad \dfrac{dx}{dt}=-32, \qquad \therefore \quad C=-32.$

$$\therefore \quad \frac{dx}{dt}=gt-32. \qquad \cdots \quad (3)$$

Integrating again with respect to t,

$$x=\tfrac{1}{2}gt^2-32t+D\,;$$

when $\quad t=0, \quad x=0, \quad \therefore \quad D=0.$

$$\therefore \quad x=\tfrac{1}{2}gt^2-32t.$$

Putting $t=17$ and $g=32$, we get

$$x=\tfrac{1}{2}\times 32\times 17^2-32\times 17$$
$$=4080 \text{ ft.}$$

$\therefore$ the required height $=4080$ ft.

Ex. 2. *A stone is dropped into a well and the sound of the splash is heard in* $7\frac{7}{10}$ *seconds. If the velocity of sound be* 1120 *feet per second, find the depth of the well.* [*U. P. 1939*]

Let h ft. be the required depth of the well, and t secs., the time taken by the stone to fall to the water surface, so that sound takes $(7\frac{7}{10}-t)$ secs. to travel the depth of well.

Then, $h=\frac{1}{2}gt^2=(7\frac{7}{10}-t)\,1120,$

or, $16t^2-1120(\frac{77}{10}-t)=0,$

i.e., $t^2+70t-7\times77=0,$ or, $(t-7)(t+77)=0,$ giving $t=7$ secs. (rejecting the negative value as inadmissible).

$$\therefore \quad h=\tfrac{1}{2}\times32\times7^2=784 \text{ feet.}$$

Ex. 3. *A and B are projected vertically upwards at the same instant with velocities* 25 *and* 200 *ft. per sec. respectively, A from the top and B from the bottom of a vertical cliff* 300 *ft. high. Find where they will meet, and the direction of their motion at the time of meeting.* [$g=32$] [*B. H. U. 1932*]

Let t secs. be the time after which the bodies meet. In this time A moves *upwards* from the top through a distance $25t-\frac{1}{2}gt^2$. In the same time B moves upwards from the bottom through a distance $200t-\frac{1}{2}gt^2$. Thus,

$$(25t-\tfrac{1}{2}gt^2)+300=200t-\tfrac{1}{2}gt^2.$$

$$\therefore \quad t=\tfrac{300}{175}=\tfrac{12}{7} \text{ secs.}$$

Hence, the point where they meet is at a height

$$200\times\frac{12}{7}-\frac{1}{2}\times32\left(\frac{12}{7}\right)^2=\frac{14496}{49}=295{\cdot}8 \text{ ft. from the}$$

bottom of the cliff.

At this instant A moves upwards with a velocity

$$25-32\times\tfrac{12}{7} \text{ ft./sec.}$$

i.e., it is really then moving downwards.

Also the motion of B upwards is then $200-32\times\frac{12}{7}$ which is positive. Thus, when they meet, A is moving downwards and B upwards.

Examples on Chapter V(a)

(*Vertical motion*)

1. A particle is projected vertically upwards from a point with a velocity of 80 ft. per sec.; find what tim elapses before it is at a height of 96 ft. When will it be 96 ft. below the point of projection ?

2. A stone is projected vertically upwards with a velocity sufficient to carry it to a height of 50 ft.; find its velocity when it is half way up.

If the projected stone rises to a height of 19·62 metres, what is its time of ascent?

3. A ball is thrown vertically upwards. Prove that it will be at half the greatest height after times whose ratio is $3+2\sqrt{2} : 1$. Prove also that the times occupied in the two halves of its ascent are approximately as 41 : 100.

4. A particle is projected upwards from the ground, and after some time it is seen at a height 21 ft. falling downwards with a velocity of 16 ft./sec. How long before this was it moving upwards through the same point and what was its velocity then? Find also the time from start to the highest point.

5. A ball is projected vertically upwards from the top of a tower with a velocity of 64 ft./sec., and reaches the foot of the tower in 6 secs.; find the height of the tower.

6. From a balloon at a height of 456 ft. above the ground, a bundle of paper is dropped. When will the bundle reach the ground if the balloon be (i) ascending, (ii) descending with a uniform velocity of 20 ft. per sec.?

7. A cricket ball is thrown vertically upwards; find through what distance it goes in the last half second of its ascent.

8. A particle after falling freely for some time under the action of gravity is observed to pass through 768 ft. in 4 secs.; how far will it fall in the next 4 secs.?

9. A particle falling under gravity describes 80 ft. in a certain second; how long will it take to describe the next 80 ft.?

10. A stone falling from the top of a house was found to take $\frac{1}{4}$ sec., in passing against the door 8 ft. high, situated at the base of the house. Find the height of the house.

11. A body falling freely from the top of a building is observed to pass through $\frac{5}{9}$ths of the height of the building in the last second of its motion. Find the height of the building.

12. A person at the top of a tower projects a body vertically upwards with a velocity of 96·6 ft./sec.; 4 seconds afterwards he lets drop a second body and both reach the ground simultaneously. Find the height of the tower and the time during which the second body was falling.

[*C. U. 1937*]

[Take $g = 32{\cdot}2$ ft./sec^2.]

13. A stone P is thrown vertically upwards with a velocity of 78 ft./sec. from the top of a high monument, and after 3 secs. another stone Q is let fall from the same point. Find when and where will P overtake Q.

14. A stone is let fall from a height of 50 ft. above the ground. At the same moment a ball is projected upwards from the ground with a velocity of 40 ft./sec. in the same vertical line. Show that they will meet midway, and find the time of meeting.

15. A ball is thrown vertically upwards with a velocity of 128 ft. per sec., and after 2 secs. another ball is projected from the same point and with the same initial velocity. When and where do they meet ?

16. P and Q are two points in the same vertical line, P being above Q. A heavy particle is projected vertically upwards from Q with a velocity which will just carry it to P and at the same time a heavy particle is dropped from rest at P. Show that when the particles meet, their velocities will be equal and opposite, and the spaces passed over by the particles will be as 3 : 1 [*C. U. 1939*]

17. A stone falling from the top of a vertical tower has descended x ft. when another is let fall from a point y ft. below the top. If they fall from rest and reach the ground together, show that the height of the tower is $\frac{(x+y)^2}{4x}$ ft.

[*C. U. 1935*]

2. A stone is projected vertically upwards with a velocity sufficient to carry it to a height of 50 ft.; find its velocity when it is half way up.

If the projected stone rises to a height of 19·62 metres, what is its time of ascent ?

3. A ball is thrown vertically upwards. Prove that it will be at half the greatest height after times whose ratio is $3+2\sqrt{2} : 1$. Prove also that the times occupied in the two halves of its ascent are approximately as 41 : 100.

4. A particle is projected upwards from the ground, and after some time it is seen at a height 21 ft. falling downwards with a velocity of 16 ft./sec. How long before this was it moving upwards through the same point and what was its velocity then ? Find also the time from start to the highest point.

5. A ball is projected vertically upwards from the top of a tower with a velocity of 64 ft./sec., and reaches the foot of the tower in 6 secs.; find the height of the tower.

6. From a balloon at a height of 456 ft. above the ground, a bundle of paper is dropped. When will the bundle reach the ground if the balloon be (i) ascending, (ii) descending with a uniform velocity of 20 ft. per sec. ?

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14. A stone is let fall from a height of 50 ft. above the ground. At the same moment a ball is projected upwards from the ground with a velocity of 40 ft./sec. in the same vertical line. Show that they will meet midway, and find the time of meeting.

15. A ball is thrown vertically upwards with a velocity of 128 ft. per sec., and after 2 secs. another ball is projected from the same point and with the same initial velocity. When and where do they meet ?

16. P and Q are two points in the same vertical line, P being above Q. A heavy particle is projected vertically upwards from Q with a velocity which will just carry it to P and at the same time a heavy particle is dropped from rest at P. Show that when the particles meet, their velocities will be equal and opposite, and the spaces passed over by the particles will be as 3 : 1 [*C. U. 1939*]

17. A stone falling from the top of a vertical tower has descended x ft. when another is let fall from a point y ft. below the top. If they fall from rest and reach the ground together, show that the height of the tower is $\frac{(x+y)^2}{4x}$ ft.

[*C. U. 1935*]

18. A ball is dropped from a point 324 ft. above the ground and after it has fallen 64 ft., another is thrown down from the same point, so that both reach the ground at the same instant. Find the initial velocity of the second ball.

19. A balloon is ascending vertically and at a height of 1500 ft. a stone is released. If the stone reaches the ground in 10 seconds, find the height through which the stone rises immediately after the release.

20. If a bomb, dropped from an aeroplane rising vertically with uniform velocity, reaches the ground in 5 secs., find the height of the aeroplane when the bomb reaches the ground.

21. A lift is ascending from the ground from rest with a uniform acceleration of 4 ft. per sec^2. At the end of 20 secs., a ball is dropped from it. Find the time that elapses before the ball reaches the ground.

22. A stone falls freely for 3 seconds, when it passes through a sheet of glass and loses half its velocity and then reaches the ground in $\frac{1}{2}$ second ; find the height of the glass above the ground.

23. A, B, C, D are points in a vertical line, the lengths AB, BC, CD being equal. If a body falls freely from A, prove that the times of describing AB, BC, CD are respectively as

$$1 : \sqrt{2}-1 : \sqrt{3}-\sqrt{2}.$$

24. A stone is dropped into a well and the sound of its striking the water is heard in $2\frac{3}{8}$ secs. If the velocity of sound be 1120 ft. per sec., find the depth of the well.

[*C. U. 1932*]

25. A stone dropped into a well reaches the water with a velocity of 80 ft. per sec, and the sound of its striking the water is heard in $2\frac{7}{12}$ secs. after it is let fall. Find the velocity of sound.

✓ **26.** A stone dropped into an empty pit of depth h is heard to strike the bottom after t secs. Prove that

$$2h\left(1+\frac{gt}{v}\right)=gt^2,$$

where v is the velocity of sound supposed so large compared with h that $\left(\frac{h}{v}\right)^2$ can be neglected. [*C. U. 1933*]

✓**27.** A rocket ascending vertically from the ground with an initial velocity of $\sqrt{2gy}$ ft. per sec. explodes when it reaches the greatest height, and the interval between the sound reaching the place of starting, and a place distant x ft. from it, is $\frac{1}{n}$th of a second. Show that the velocity of sound is $n\left(\sqrt{x^2+y^2}-y\right)$ ft./sec.

28. Three particles are simultaneously projected vertically upwards from heights h_1, h_2, h_3 above the ground, with velocities u_1, u_2 and u_3 respectively, and all of them reach the ground at the same instant. Prove that

$$u_1(h_2-h_3)+u_2(h_3-h_1)+u_3(h_1-h_2)=0.$$

✓ **29.** A particle thrown vertically upwards takes t secs. to rise to a height h and t' secs. is the subsequent time to reach the ground again. Show that $h=\frac{1}{2}gtt'$. [*C. P. 1963*]

30. From an aeroplane rising vertically with uniform acceleration f, a ball is dropped ; 4 secs. after this another ball is dropped from the aeroplane. Show that the distance between the two balls 2 secs. after the second ball is dropped is $16(g+f)$.

31. Two particles are projected, from the same point at the same instant with the same velocity, one vertically upwards, the other vertically downwards. The first takes t_1 secs. to reach the ground, and the second t_2 secs. to reach it. Prove that either of them falling freely downwards from the same point reaches the ground in $\sqrt{t_1 t_2}$ secs.

✓ **32.** A man in a lift, ascending with an acceleration f ft./sec^2, throws a ball vertically upwards with a velocity v ft. per sec. relatively to the lift, and catches it again in t secs. ; show that $f+g=2v/t$. [*C. P. 1964*]

33. If a particle takes t seconds less time and acquires a velocity v ft./sec. more at one plane on the earth's surface than at another in falling freely through the same height, show that the geometric mean of the numerical values of g at the two places is v/t.

34. The heights of a particle projected vertically upwards are x, y, z at times t_1, t_2, t_3 respectively measured from the instant of projection ; prove that the value of g is

$$2\left[\frac{x_1(t_2-t_3)+x_2(t_3-t_1)+x_3(t_1-t_2)}{(t_2-t_3)(t_3-t_1)(t_1-t_2)}\right]$$ [*C. H. 1967*]

ANSWERS

1. 2 secs. and 3 secs. ; after 6 secs. from start.
2. 40 ft /sec. ; 2 secs. **4.** 1 sec., 16 ft./sec., $1\frac{1}{2}$ secs.
5. 192 ft. **6.** (i) 6 secs. (ii) $4\frac{3}{4}$ secs. **7.** 4 ft. **8.** 1280 ft.
9. $(\sqrt{14}-3)$ secs. **10.** $20\frac{1}{4}$ ft. **11.** 36 ft. **12.** 257·6 ft. ; 4 secs.
13. After 5 secs. from the starting of Q, at a depth of 400 ft. from the starting-point. **14.** $1\frac{1}{2}$ secs from start.
15. 8 secs. after the second ball is projected, at a height 240 ft.
18. 89·6 ft./sec. **19.** $1\frac{9}{16}$ ft. **20.** 400 ft. **21.** 10 secs.
22. 28 ft. **24.** 100 ft. **25.** 1200 ft./sec.

5·6. Motion on a smooth inclined plane.

Let XYZ be a smooth inclined plane, inclined at an angle α to the horizon. P being any point on it, let

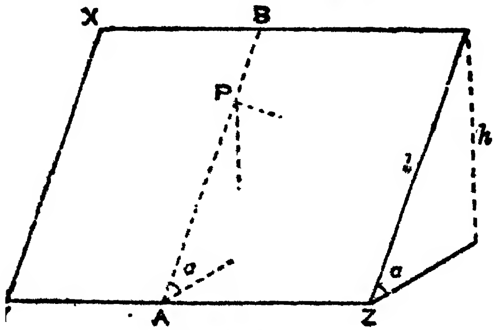


Fig. (i)

BPA be the section of it by the vertical plane through P containing the normal to the plane. This line BPA, which is evidently perpendicular to the line of intersection YZ of the inclined plane with the horizon, is defined as the line of greatest slope through P along the plane, for, from Geometry, it is easily proved that the inclination of this line to the horizontal plane, which is clearly a, is greater than that of any other line through P on the plane.

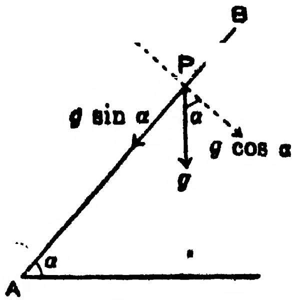


Fig. (ii)

Figure (ii) shows the plane of the section in question.

Now, if a particle be situated at P on the plane, the vertical acceleration due to gravity, g, with which the particle would fall freely in absence of the plane, may be broken up by the principle of parallelogram of accelerations, into two components, one $g \cos a$ normal to the plane, and the other $g \sin a$ along the line of greatest slope PA on the plane. The plane prevents any motion perpendicularly through it by producing a normal reaction which nullifies the effect of the normal component of acceleration $g \cos a$. Hence, *the only component of acceleration with which the particle will move on the plane is $g \sin a$ down the plane along the line of greatest slope.*

Any problem, therefore, of rectilinear motion of a particle either upwards or downwards along a line of greatest slope on a smooth inclined plane may be worked out with the help of the usual formulæ for uniformly accelerated motion in a straight line by replacing f by $+g \sin a$ or $-g \sin a$ according as the *downward* or the *upward* direction is taken as positive.

Note. In considering the rectilinear motion on an inclined plane along the line of greatest slope, the length of this line of greatest slope will be referred to as "the length of the inclined plane". Also the height of the topmost point of the plane is called "the height of the

plane." If now h and l be the height and length of an inclined plane of inclination a to the horizon, it is evident that $\sin a = \frac{h}{l}$.

5·7. Body sliding down a plane.

Let a body be allowed to slide down from the top of a smooth inclined plane of length l and inclination a to the horizon. Taking the downward direction as positive, the acceleration down the plane is $g \sin a$. If t be the time to slide down, and v be the velocity acquired on reaching the bottom,

we have

$$l = \tfrac{1}{2}g \sin a . t^2, \text{ or } t = \sqrt{\frac{2l}{g \sin a}},$$

and $$v^2 = 2g \sin a . l, \text{ or } v = \sqrt{2gl \sin a}.$$

Cor. If h be the height of the plane, since $\sin a = \frac{h}{l}$, we can write $v = \sqrt{2gh}$ showing that if from the same height, particles be allowed to slide down different inclines, the velocity on reaching the ground is the same in all cases, and equal to that acquired in falling freely through the same height.

5·8. Body projected up an inclined plane.

Let a body be projected with a velocity u from the bottom of an inclined plane of inclination a to the horizon, along the line of greatest slope.

Taking the upward direction along the line of greatest slope as positive, the acceleration along it is $-g \sin a$. Let L be the length described by the body when it is at the greatest height attainable, *i.e.*, when its velocity is zero, and let T be the corresponding time.

Then, $$0 = u - g \sin a . T,$$

and $$0 = u^2 - 2g \sin a . L.$$

$$\therefore \quad T = \frac{u}{g \sin a}, \quad L = \frac{u^2}{2g \sin a}.$$

Cor. 1. If H be the vertical height from the ground in the above case when the velocity of the particle is zero, we get

$$H = L \sin \alpha = \frac{u^2}{2g}.$$

Hence, if different bodies be projected upwards along different inclines with the same starting velocity, they rise to the same height in every case, the height being the same as attained by a particle projected vetically upwards with the same velocity.

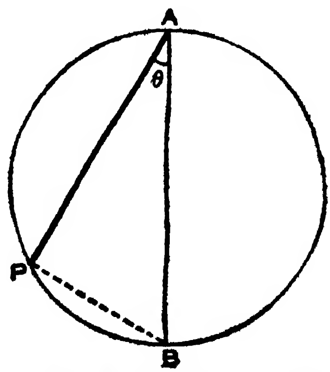
Cor. 2. After reaching the greatest height attainable on the plane, the particle will again slide down, and just as in the case of vertical motion we can show in this case also that

time of rise = time of fall,

and *the velocity on reaching the starting point again = the initial velocity of projection.*

59. Motion down a chord of a vertical circle.

The time taken by a body to slide down any smooth chord of a vertical circle, starting from rest at the highest point of the circle, is constant.



Let AB be the vertical diameter of a vertical circle, so that A is the highest point of the circle. Let AP be any chord of the circle, through A, assumed perfectly smooth. Now θ being its inclination to the vertical diameter AB, $90^\circ - \theta$ is its inclination to the horizon, and so acceleration of a body sliding down it is $g \sin (90^\circ - \theta) = g \cos \theta$.

Also AB being a diameter of length d say, $\angle APB$ is a right angle, and so $AP = d \cos \theta$.

Now T being the time of sliding down AP, starting from rest at A,

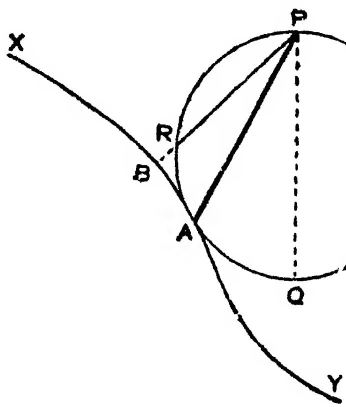
$$d \cos \theta = \tfrac{1}{2} g \cos \theta . T^2$$

whence $T=\sqrt{\dfrac{2d}{g}}$, a constant independent of θ, and so same for all chords.

N. B. θ being the inclination of an incline to the *vertical*, acceleration down the incline is $g \cos \theta$.

Note. It can be shown in exactly a similar way as above that *the times of sliding down from rest along all chords of a vertical circle ending at the lowest point are equal.*

5·10. Line of quickest descent.



Suppose P is a given point, and XY a given curve in a vertical plane through P. If now a particle is to slide down from P along a straight line to reach XY, that line from P to XY along which the time of sliding is the least is defined as the line of quickest descent from P to XY.

To construct such a line, if we assume the circle PAQ to be drawn having its highest point at P, and touching the given curve XY at some point A say, then PA is the line of quickest descent from P to XY.

For, if PB be any other line from P to XY meeting the circle at R, then by Art. 5·9 the times of sliding down PR

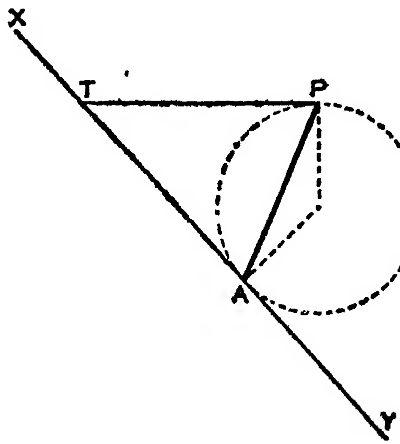
and PA are equal, and so the time of sliding from P to B along PBB is longer than that from P to A along PA.

It is evident then, that the line of quickest descent from P to XY is not necessarily the same as the line of shortest length drawn from P to the curve XY.

We investigate below two particular cases of construction of the line of quickest descent.

(*i*) *Line of quickest descent from a given point P to a given straight line XY in the same vertical plane.*

Through P draw the horizontal line PT in the plane PXY to meet XY at T, and cut off TA downwards along XY making $TA = TP$. Then PA is the required line of quickest descent from P to XY.

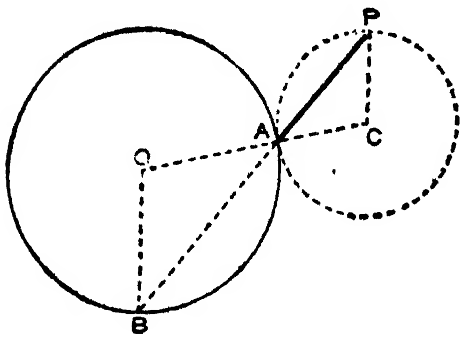


For, since $TA = TP$, a circle can be drawn to touch PT and AT at P and A respectively, and this circle, having the horizontal line PT as the tangent at P, is clearly the circle with P as the highest point, and as it touches XY at A, the line PA, as proved above, is the line of quickest descent.

(ii) ***Line of quickest descent from a given point P to a given circle in the same vertical plane.***

O being the centre of the given circle, let OB be drawn vertically downwards to meet the circle at B. Join PB, and let it meet the circle at A. Then PA is the required line of quickest descent from P to the circle.

For PC being drawn vertically downwards, parallel to OB, and OA produced intersecting PC at C, it is easily proved from geometry that $CP = CA$. Hence, the circle with centre C and radius CP will touch the given circle at A. Now PC being vertically downwards, P is the highest point of this circle which touches the given circle at A.



Thus, as proved before, PA is the required line of quickest descent.

5·11. Illustrative Examples.

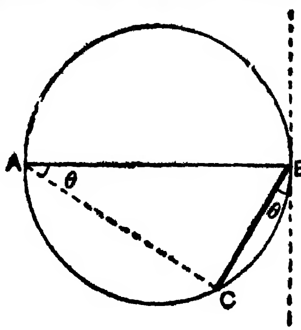
Ex. 1. *With what velocity must a particle be projected up a plane, 10 ft. in length and inclined to the horizon at an angle of 30°, so as to reach the top in one second?* [*C. U. 1912*]

Let u ft./sec. be the velocity of projection. The acceleration of the particle down the plane is $g \sin 30° = 32 \times \frac{1}{2} = 16$ ft /sec² in this case.

Hence, from the given conditions,

$$10 = u \times 1 - \tfrac{1}{2} \times 16 \times 1^2$$
$$= u - 8, \qquad \text{or,} \quad u = 18 \text{ ft./sec.}$$

Ex. 2. *If a chord is drawn from one end of the horizontal diameter to any point of a vertical circle, show that the time that a particle would take in sliding down that chord would vary as the square root of the tangent of the inclination of the chord to the vertical.* [*U. P. 1938*]



θ being the inclination to the vertical of the chord BC of a vertical circle, drawn from the extremity B of the horizontal diameter AB, the acceleration down the chord is $g \cos\theta$. Hence, t denoting the time taken to slide down BC, starting from rest at B,

$$BC = \tfrac{1}{2} \cdot g \cos\theta \cdot t^2,$$

or, $$d \sin\theta = \tfrac{1}{2} \cdot g \cos\theta \cdot t^2.$$ [where $d = AB$ and $\angle BAC = \theta$ easily]

Thus, $$t = \sqrt{\frac{2d}{g} \tan\theta}$$

i.e., $$t \propto \sqrt{\tan\theta}.$$

Ex. 3. *A cyclist rides up an incline of* 1 *in* 64 *with a uniform acceleration of* 4 *ft./sec*^{2} *starting from rest at the foot. After* $\frac{1}{2}$ *a minute he meets another cyclist descending from the top without pedalling, starting simultaneously with him. After how much more time will the first cyclist arrive at the top ?*

By an incline of 1 in 64 is meant that the incline is such, that in moving a distance of 64 along it, one rises a height 1 ; in other words, this means that the sine of the angle of inclination of the plane to the horizon is $\frac{1}{64}$.

The acceleration of the second cyclist down the plane is therefore $\frac{1}{64} g = \frac{1}{2}$ ft./sec^2.

In half a minute (*i.e.*, 30 secs.) the second cyclist has descended from the top, a distance $\frac{1}{2} \cdot \frac{1}{2} . 30^2 = 225$ ft.

The velocity of the first cyclist then is $v=4.30=120$ ft./sec. Thus, t seconds denoting the time taken by him to reach the top from this instant,

$$225=120t+\tfrac{1}{2}.4.t^2,$$

$$\text{or,}\quad 2t^2+120t-225=0.$$

$$\therefore\quad t=\frac{-120+\sqrt{120^2+8.225}}{4}$$

$$=\tfrac{15}{2}(3\sqrt{2}-4)=1{\cdot}82 \text{ seconds nearly.}$$

Examples on Chapter V(b)

(*Motion on an inclined plane*)

1. A train running at the rate of 60 miles per hour shuts off steam on reaching the foot of an incline of 1 in 120. How far will it run up the incline ?

2. The length of a plane inclined at an angle 30° to the horizon is 150 yds. A body is projected up the plane from its foot with a velocity just sufficient to carry it to the top. Show how to divide the length of the plane into three parts which are traversed by the body in equal times.

3. A particle is allowed to slide down an inclined plane from its top, and after describing $\frac{3}{4}$ths of the length, passes a second particle which was simultaneously projected up the plane from the foot. What fraction of the total height of the plane does the second particle rise ?

4. An engine rises up an incline of 1 in 20 with a uniform acceleration of 2 ft./sec^2, starting from rest at the foot. After a certain distance the steam is shut off, and the impetus just carries the engine to the top. If the length of the incline be $4\frac{1}{2}$ miles, find where the steam was shut off.

5. If two vertical circles touch each other at (i) their highest points, (ii) their lowest points, and a straight line be drawn from this point cutting the circles, show that the time of sliding from rest down the part between the circumferences supposed smooth, is constant.

6. A number of straight lines are drawn in a vertical plane through a fixed point O, and particles are allowed to slide down these, all starting simultaneously at rest from O. At any instant t, show that they lie on a circle of radius $\frac{1}{4}gt^2$.

7. A particle starts from rest from the top of a smooth inclined plane of a given base. Show that the time of fall is least when the inclination of the plane to the horizon is $45°$. [*C. U. 1938*]

8. Two smooth inclined planes of the same altitude and of elevations α and β stand back to back. A body projected up the first plane from its foot along the line of greatest slope with a velocity u, ascends it, and without losing any velocity at the turn, descends the second plane. Find its velocity at the foot of the second plane.

9. Two particles slide down two straight lines in the same vertical plane, at right angles to one another, starting simultaneously from rest from their point of intersection. Prove that the distance between them at any time will be equal to the distance either would have descended vertically in that time.

10. Two heavy particles begin to slide at the same instant from the common vertex of two smooth inclined planes. Prove that the line joining them moves, remaining always parallel to itself.

11. One side of a triangle is vertical. If the times of fall from rest down the other sides are equal, prove that the triangle is either isosceles or right-angled.

12. A parabola has its axis vertical and its vertex at the lowest point. Prove that the time of descent of a particle down any smooth chord to the lowest point is equal to that of falling vertically to the horizontal line which is at a depth below the vertex equal to the latus rectum.

13. In a vertical circle two chords are drawn from an extremity of a horizontal diameter, subtending angles α and 2α at the centre; if the times of sliding down these chords be as $1 : n$, show that $\sec \alpha = n^2 - 1$.

14. From a given point on an inclined plane smooth grooves are cut along different directions up to the base line. Prove that the times of sliding down these from the given point are proportional to the lengths of the grooves.

15. A particle sliding down an inclined plane is observed to pass over two consecutive equal distances of 3 feet in $\frac{1}{2}$ and $\frac{1}{4}$ sec. respectively. Find the inclination of the plane to the horizon.

16. A particle slides from rest down a smooth plane inclined at 30° to the horizon. Find the position of a length of 80 feet on its path which is passed over by the particle in one second.

17. Two particles are allowed to slide down an inclined plane from the same point with an interval of one second between the times of starting. Show that their distances from each other at the ends of 1, 2, 3, 4,... seconds are as 1, 3, 5, 7,......

18. A tangent at any point P of a circle meets the tangents at the extremities of a vertical diameter AB in C, D respectively. If t_1 and t_2 be the times of sliding from rest down CP and PD respectively, then

$$\frac{t_1}{t_2} = \frac{\text{chord } AP}{\text{chord } BP}.$$

ANSWERS

1. $2\frac{2}{3}$ miles. **2.** $83\frac{1}{3}$ yds., 50 yds., $16\frac{2}{3}$ yds.

3. $\frac{1}{3}$. **4.** 2 miles from the bottom. **8.** u.

15. 30°. **16.** The length begins after a distance of 162 feet from the starting point.

CHAPTER VI

LAWS OF MOTION

6·1. So far we have been dealing with kinematics only, that is, we have been considering only different kinds of motions, and the effects thereof on the position etc. of particles without entering into the causes which produce these motions. In this chapter we shall discuss the relations between the forces which produce motions and the types of motion produced thereby, which is Dynamics proper. These relations are based wholly on the three well-known laws of Newton. They run as follows :

Newton's Laws of Motion.

First Law—*Every body continues in its state of rest or of uniform motion in a straight line except in so far as it be compelled by any external impressed force to change that state.*

Second Law—*The rate of change of momentum is proportional to the impressed force, and takes place in the direction in which the force acts.*

Third Law—*To every action, there is an equal and opposite reaction.*

These laws are more or less axiomatic, and have been formulated on the basis of common-sense and experience. They have been verified indirectly for bodies of ordinary size (within the limits of experimental error) by experimenting on the results deduced from the laws. Moreover, the exactness with which the positions and motions of bodies on earth, as also of celestial bodies like sun, moon and planets have been predicted from calculations based on these laws lend the strongest support towards the assumption of the laws as fairly true.

6·2. Explanation and Illustration of the First Law.

This law is practically embodied in the very definition of Force. A force has been defined to be that which changes (or tends to change) the state of rest, or of uniform motion of a body. This means that if a body be at rest, anything which changes its state of rest is a force. Thus in absence of a force, a body at rest will continue to be at rest. Similarly, if a body be in motion, anything which changes its motion is a force, and so in absence of a force, a body in motion will have its motion unaltered, or in other words, it will continue to move uniformly. These two statements together form the first law.

The tendency of a body to continue in its state of rest or of uniform motion as the case may be, in absence of any external force, is defined to be the property of *inertia.* Hence, Newton's first law of motion is also referred to as the **Law of Inertia.**

The first part of the law, namely that a body at rest does not move of its own accord unless compelled by an applied force to do so, is a matter of common experience, and requires no particular illustration. The second part of the law, however, that a body in motion will continue to move uniformly forever in absence of any applied force, is a matter, which strictly speaking, cannot be experienced in practice from direct observations, for in the practical world we can never make a moving body absolutely and continuously free from the influence of external forces. The tendency however of moving bodies to continue their motions when unhampered by external forces may be conceived from the following illustrations :

When a man alights from a rapidly moving car, his feet, coming in contact with the rough ground, are brought to rest by the friction of the ground ; but the upper part of the body, which was sharing the motion of the car, having a tendency to continue the motion, the man generally falls down.

If a galloping horse suddenly stops, the rider on its back is in danger of being thrown over the horse's head.

When a passenger is sitting sideways on a tram-car, as the car starts from rest, the part of his body in contact with the seat moves forward with the car, while the upper part of his body, having a tendency to continue in its position of rest, is generally thrown backwards. Similarly, when the car stops from motion, the upper part of his body leans forward.

A circus rider on a running horse suddenly jumps up and passing through a ring of fire, suitably placed above, again alights just on the back of the horse. Here, the forward motion which he was sharing with the horse tends to continue practically unchanged (the resistance being negligible, there is practically no horizontal force to affect the horizontal motion) all the time he is rising and falling due to gravity. So the distance moved forward by him during the interval is exactly equal to that moved over by the horse.

When a heavy piece of stone is hanging by a fine thread, the position in which the system rests is that in which the string is vertical, and in this position the forces acting on the body (namely the tension of the string, and the attraction of the earth) balance one another, so that there is no resultant force left on the body. In absence of any external force, the body continues to be at rest in this position. If now the stone be pulled to one side and let go, the forces on the body not balancing one another, a motion will be produced. As the system comes to the former vertical position, there will be no resultant force on the body; but in absence of a force, the body, which has already acquired some motion, does not stop, but tends to continue its motion, and moves over to the other side. Thus, in the same position, where no resultant force acts on the body, in one case when the body is at rest, it continues its state of rest, while in the other case, when the body is in motion, it continues its state of motion.

If a ball on a plane ground be given a motion, it is observed that after a short time the motion of the ball ceases, and it comes to rest on account of the friction of the ground. If however the ground be made smooth, for instance

a long smooth track on ice accumulating on the ground in cold countries during winter be prepared by rubbing on it with a piece of ice, and a smooth body be allowed to slide over it, the motion continues for a pretty long time. Though ultimately, on account of air resistance and other forces the body comes to rest, it gives us a good idea as to the fact that if it were possible to make the body free from the influence of all resisting forces, the motion once generated, would continue for ever.

The first law gives us **a qualitative test of the existence of a force,** for whenever we see a body remaining at rest, we say that there is no resultant force acting on it. Similarly, *when we find a body moving with a uniform velocity, we conclude that there is no resultant force acting on it.* On the other hand, when the motion of a body is changing, either in magnitude, or in direction, there must be some force acting on it.

6·3. Second Law of Motion : Momentum.

Momentum*—*The momentum of a moving particle at any instant is the product of its mass and its velocity at that instant.*

As velocity has got a definite direction at any instant, the momentum of a moving particle has also got a magnitude and a direction, and is thus a vector quantity.

The second law of Newton aims at a **quantitative measurement of a force** (§ *6·4*). Now, the effect of a force on a body is generally to generate motion, or to produce a change of motion. Hence to make an idea as to the magnitude of a force we are to notice the motion generated (*i.e.*, the change of motion produced) by it in a body, in a given time, say one second. If in the same body, in the same time, another force generates a greater motion, that force is greater than the former. If the motion generated in the

* This is sometimes referred to as *linear momentum* to distinguish it from *angular momentum* (moment of momentum) defined in books on Advanced Dynamics.

latter case is double of that in the former case, it is common-sense that the latter force is double the former. Again, if we consider two different bodies, one heavier than the other, then it is common experience that the same force applied to these bodies for the same time will not generate the same motion in the two ; the motion generated in the heavier body will be noticed to be less. Also, to generate equal motions in the two bodies in the same time, the force required in the case of the heavier body is greater. Thus, to make an estimate of the measure of a force we are to note, not only the velocity generated by its action on a body in a definite time (preferably unit time), but also to consider the mass of the body ; the measure of the force depends on the product of the mass and the velocity generated in a given time, *i.e.*, on the momentum generated. Hence is the necessity of defining such a mathematical quantity as momentum. The effect of a force is to generate momentum. In case of a body of constant mass, the effect of a force is to change its velocity, *i.e.*, to generate acceleration in it.

Note. It may be noted that for a body where mass continually increases (for example, a falling rain-drop on which aqueous vapour is continually accumulating and making it bigger in size), even to keep its velocity unchanged, a force will be required, for the momentum increases in this case. In absence of a force, its velocity will gradually diminish but its momentum will remain unchanged.

6·4. *To deduce the formula* $\mathbf{P = mf}$.

Let a force of which the measure is P be acting continuously on a particle of mass m, and let v be the velocity and f the acceleration of the particle at any instant during the action of the force.

Then by Newton's second law of motion,

$P \propto$ rate of change of momentum of the particle

i.e., $\propto$ rate of change of mv

i.e., $\propto m \times$ rate of change of v

[provided the mass of the body is unchanged throughout the motion]

i.e., $\propto mf$.

Hence, $P = K.mf$ where K is a constant. We have not as yet defined any unit for measuring a force. Let a unit of force be chosen to be that amount of force which acting continuously on a unit mass, produces in it a unit acceleration.

Then, in the result $P = Kmf$, when $m = 1$, and $f = 1$, by our choice, $P = 1$.

Hence. $K = 1$. Thus, when expressed in such units,
$P = mf$.

Note. Such a unit of force is defined to be an **absolute** (or **dynamical) unit of force.**

Analytically, $P = K \dfrac{d}{dt}(mv) = Km \dfrac{dv}{dt}$,

assuming m remains constant throughout the motion.

$\therefore \quad P = Kmf.$

Now, choosing the units suitably we have $K = 1$, as above.

$\therefore \quad P = mf.$

65. F.P.S. and C.G.S. absolute units of Force.

Poundal—*A poundal is that amount of force which acting continuously on a mass of one pound, produces in it an acceleration of one foot per second per second.*

It is used as a unit for measuring forces in F.P.S. system.

Dyne—*A dyne is that amount of force which acting continuously on a mass of one gram, produces in it an acceleration of one centimetre per second per second.*

It is used as a unit for measuring forces in C.G.S. system.

It may be noted that the formula $P = mf$ will be satisfied (i) when P is expressed in poundals, m in pounds and f in ft./sec^2, or, (ii) when P is expressed in dynes, m in grammes and f in cms./sec^2. A poundal and a dyne are *absolute* units of force.

Relation between a poundal and a dyne :—

Since 1 foot = 30·48 cms. nearly,

and 1 lb. = 453·6 grams,

1 poundal = 30·48 × 453·6 dynes

= 13800 dynes roughly (in round figure).

6·6. Weight—*The weight of a body is the force with which the earth attracts the body.*

Now it is known that due to the attraction of the earth at any point on it, everybody moves towards the earth with a uniform acceleration g. Hence, if m be the mass of a body, then W being its weight, that is the force exerted on it due to earth's gravitation, the acceleration produced in the body due to this force being g, we have (from the formula $P = mf$), $\mathbf{W = mg}$ in absolute units (poundal or dynes as the case may be).

Thus, expressing g in ft.-sec. units and cm.-sec. units respectively,

weight of a mass of 1 *lb.* = 32 *poundals roughly,*

and *weight of a mass of* 1 *gm.* = 981 *dynes roughly.*

Gravitational units of force—The weight of 1 lb. as also the weight of 1 gm. are sometimes used as units for measurement of forces. These are referred to as gravitational units of forces, as they depend on the value of 'g', the acceleration due to earth's gravitation. As the value of g depends on the position on the earth, and changes (though very slightly) from place to place on the surface of the earth, the gravitational units also vary slightly from place to place. *Roughly, dividing a force in poundals by* 32 *it is expressed in lbs. weight, and similarly by dividing a force in dynes by* 981 *it is expressed in grammes weight.*

We notice therefore that whereas the absolute units of force, poundals or dynes, have nothing to do with the position on the surface of the earth and are therefore, invariable, the gravitational units of force depend on the value of g, and so on the place on earth where it is used.

6·7. Distinction between mass and weight.

The mass of a body is the quantity of matter in the body. The weight of the body on the other hand is the force with which the earth attracts the body. Whereas the former is an intrinsic property of the body itself, and has nothing to do with any other body, the latter depends, not on the body alone, but also on the earth, or on the position of the body with respect to the earth. At different positions on the surface of the earth the force of attraction of the earth on the same body is different and so the weight of the body alters from place to place. If it were possible to take the body to the centre of the earth, the resultant attraction of the earth on it, from symmetry, would be nil, and so the body would have no weight. But the mass of the body all along remains unchanged, so long as no material part of it is removed.

In common language we loosely use the term weight to mean its mass and speak of the weight of a body to be 10 lbs. or 15 gms., which is an incorrect statement. The confusion arises due to the fact that at a given place on the surface of the earth, the weights of bodies being proportional to their masses, two bodies having their masses equal will have their weights also equal *i.e.*, they will be equally attracted by the earth, and this is made use of for determination of the mass of a body by weighing. We place the body in question on one pan of a balance and standard bodies with known masses (engraved on them) on the other, until the beam of the balance is horizontal, when we know that the forces of attraction of the earth on the two sides are equal, and hence the masses on either side are also equal. The mass of the body in question therefore becomes known.

The mass of the body being the same as that of standard bodies of known total mass 10 lbs. say, we should correctly speak of the weight of the body to be equal to the weight of a body of mass 10 lbs. or briefly, *the weight of the body is equal to* 10 *lbs. weight.*

6·8. Spring-Balance.

The change in the weight of a body referred to above from place to place on the surface of the earth cannot be

detected by weighing with an ordinary balance. For suppose that the weight of a body is determined at one place by weighing. The mass of the body then is equal to that of the standard weights used on the other pan when the weights on the two sides balance. If now we proceed to some other place on earth where the value of g is different, the weight of the body in question (*i.e.*, the force of attraction of earth on it) is altered. But simultaneously the weight of the standard weights used will also alter, and the masses on the two sides being the same, the weights of the two sides will balance here also, and thus the observed weight as determined from the writings on the standard weights used will be the same as before. For detecting the change in the weight of a body from place to place on earth, a spring-balance may be used.

A *spring-balance* essentially consists of a spiral spring, to the lower end of which a pan or hook is attached. It is suspended from the top of a graduated vertical stand. When any weight is placed on the pan, or attached to the hook, the spring is lengthened. A pointer is attached near lower part of the spring which shares the upward or downward motion of the spring, and the graduation of the vertical stand against which the pointer points when a particular body is placed on the pan determines the weight of that body. It is evident that the same body placed on the pan of the spring-balance at different places on earth, being differently attracted by earth, the elongations of the spring will be different, and so different weights will be indicated by the pointer.

6·9. Principle of Physical Independence of Forces.

The second part of Newton's second law of motion states that the effect on a body (change of momentum, or acceleration in a body of constant mass), due to a force, is produced

along the line of action of the force. The implication is that the effect of a force on a body will be produced in its own direction under all circumstances, whether the body is at rest or has an initial velocity in some other direction, or is acted on by other forces.

If a body has an initial velocity in any direction, and is simultaneously acted on by several forces in different directions, the law implies that *each force produces an acceleration in its own direction quite independently of the presence of the others,* and the actual accelaration of the body will be the resultant of the simultaneous accelerations severally produced by the forces. The actual motion of the body will be obtained by considering the initial velocity of the body together with the resultant acceleration obtained as above, simultaneously being possessed by the body.

This principle, which is embodied in the second part of Newton's second law of motion is known as the *principle of physical Independence of Forces.*

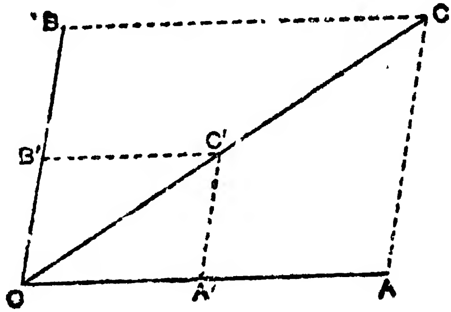
As an illustration, it may be noticed that if a stone be dropped by a passenger inside a compartment of a railway train from any height, it will strike the same point of the floor, whether the train be at rest, or be rapidly moving (with uniform velocity), and the time of fall will be the same in both cases. This shows that the vertical motion, which is due to the force of gravity, is unaffected by the initial horizontal motion which the stone possesses in the second case in common with the train, whereas the horizontal displacement of the stone during its fall is the same as that of the train, so that the initial horizontal motion of the stone is unaffected by the vertical force of gravity, which only produces effect in its own direction.

6·10. Parallelogram of Forces.

If a particle be acted on by two forces represented in magnitude and direction by two given straight lines drawn from a point, their resultant is a single force represented in magnitude and direction by the diagonal through that point

of the parallelogram drawn with the given straight lines as adjacent sides.

Let a particle of mass m be acted on by two forces represented in magnitude and direction by the lines OA and OB. From Newton's second law of motion, the effect of these two forces simultaneously acting on the particle will



be to produce two simultaneous accelerations in their respective directions, say OA' and OB', where

$$OA = m.OA' \text{ and } OB = m.OB'.$$

Now by parallelogram of accelerations, these two simultaneous accelerations OA' and OB' are equivalent to a single acceleration OC', where OC' is the diagonal of the parallelogram $OA'C'B'$. Again, this single acceleration OC' of the particle m might be produced by a force OC in this direction given by

$$OC = m.OC'.$$

Thus, the joint effect of the two forces OA and OB acting on the particle is the same as that of the single force OC. Hence, OC represents the resultant force.

Now join AC and CB. Since $\frac{OA}{OA'} = m = \frac{OC}{OC'}$, AC is parallel to $A'C'$ and accordingly parallel to OB' or OB. Similarly BC is parallel to OA. Hence, $OACB$ is a parallelogram, and OC, which represents the resultant force, is its diagonal. Thus the parallelogram of forces is established.

6·11. Remarks on the Third Law of Motion: Illustrations.

Newton's Third Law of Motion gives us an insight as to how forces act in nature. It asserts that forces never exist singly, but always appear in pairs. A force may be exerted either by direct contact, as when one body presses against another or pulls it; or it may be exerted as an attraction or repulsion between two bodies from a distance, as in the case of earth's gravitation; or it may be of the nature of a passive resistance like friction etc. But whatever be the nature of the force, it always requires two bodies or two parts of body for the exertion of forces, and corresponding to a force exerted by one part on the other, there is always an equal force exerted by the second on the first. One of these being referred to as the *action*, the other is called the *reaction*. The two together are termed *stress* between the two bodies. Every individual force acting anywhere being either an action or a reaction is merely one of the two aspects of the complete action between two bodies (or two parts of a body), and it must have its equal and opposite counterpart somewhere else.

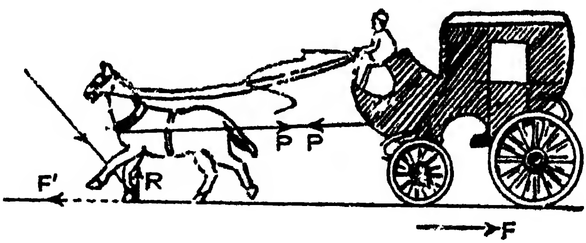
Beginners may question, "If corresponding to any force acting on a body, there is always an equal and opposite reaction, why should a body move at all?" They must bear in mind that the action and reaction do not act on the same body (or on the same part of the body). In attacking a mechanical problem it is essential to begin by fixing upon the particular portion whose motion we are going to consider, and then see which of the pair of equal and opposite actions act on this portion. Some illustrations will make this point clear.

1. When a body rests on a table, it exerts a pressure on the table on account of its weight. It itself however experiences an upward force of support (or reaction) from the table which balances its weight and thus keeps it in position. If the reaction were not exactly equal to the weight, the body would move. If the table be removed, the supporting force (or reaction) being removed, the body falls

on account of its weight ; but simultaneously the pressure on the table is removed. Here the action (in the form of pressure) is exerted on the table, and the reaction (in the form of the upward force of support) acts on the body.

2. When a magnet attracts a piece of iron, the iron also attracts the magnet with an equal force. This may be verified by placing a small magnet and a piece of iron on a smooth table sufficiently near to one another, when it will be noticed that if the iron be held fixed and the magnet be free, it will move towards the iron, just as when the magnet is held and the iron kept free, the latter moves towards the former.

3. Next let us consider the typical case of a horse dragging a carriage. As the horse pulls the carriage forwards, the carriage pulls the horse backwards, with an equal force. How is it that they ever get into motion ?



The horse in attempting to draw the cart strikes the ground obliquely with its hoofs, thereby exerting a force on the ground. The ground consequently offers an equal and opposite counterforce. Let F' be the horizontal component of force received by the horse from the ground.

Let P be the pull exerted by the horse on the carriage through the connecting string. The horse gets an equal backward pull from the carriage. Let F be the total resisting force on the carriage due to friction of the ground, etc.

Now considering the carriage alone, it moves provided $P > F$. If m be the mass of the carriage, its acceleration f will be given by

$$P - F = mf. \qquad \cdots \quad \text{(i)}$$

Considering the horse separately, it will be able to move provided $F' > P$, and if m' be the mass of the horse, the acceleration of the horse being the same as f,

$$F' - P = m'f. \qquad \cdots \quad \text{(ii)}$$

Considering the horse and carriage together as the system, P, P now form equal and opposite internal forces, in the system, which cancel one another, and the resultant force on the system is

$$F' - F = (m + m')f \qquad \cdots \quad \text{(iii)}$$

which can as well be obtained from (i) and (ii). Thus the system moves, provided only the horse can strike the ground with such a force that the forward horizontal component of the reaction received from the ground, namely F' is $> F$, the resistance to motion of the carriage. In this case, P will adjust itself to be equal to $\dfrac{m'F + mF'}{m + m'}$, as required by (i) and (ii), and f will be given by (iii).

6·12. Pressure of a body resting on a moving horizontal plane.

Let a body of mass m be placed on a horizontal plane.

Case I. When the horizontal plane is rising vertically upwards with an acceleration f.

The mass m on the plane rises along with the plane with the same acceleration f and the force which causes it to rise is supplied due to its contact with the plane in the form of the upward reaction of the plane. Assume this force of reaction to be R.

The downward force of gravity on the body is its weight mg. Hence, the resultant upward force on it is $R - mg$, and this produces the acceleration f in the body. Hence, by Newton's second law of motion,

$$R - mg = mf,$$

or,

$$R = m(g + f).$$

Now by Newton's third law of motion, the pressure exerted by the body on the plane is equal and opposite to the reaction of the plane on it. Hence, for the plane rising with acceleration f, the pressure exerted by a body of mass m placed on it, is

$$\mathbf{R = m(g+f).}$$

Case II. When the horizontal plane is descending vertically downwards with an acceleration f.

The mass on the plane also descends with the same acceleration f, and hence the resultant force on it is downwards now, so that the upward reaction on it due to its contact with the plane is less than its weight mg.

The equation for downward motion of the body separately in this case is

$$mg - R = mf,$$

and so $$R = m(g-f).$$

The pressure exerted by the body on the plane, being equal and opposite to the above reaction on the body due to the plane, is

$$\mathbf{R = m(g-f)}$$

in this case.

The above also explains how a man on a rising lift feels himself heavier, and one on a descending lift feels himself lighter than his actual weight.

Cor. *If the plane be at rest, or is moving upwards or, downwards with a uniform velocity*, f being zero, the pressure on the plane exerted by the body is $\mathbf{R = mg}$, just equal to the weight of the body.

It may be noted that even though the plane may rise, if its upward velocity be gradually diminishing, the upward acceleration is negative, and the pressure on the plane will be less than the weight mg. Similarly though the plane may descend, if its downward velocity gradually diminishes, its acceleration will be positive upwards, and the pressure

on the plane in this case will be greater than the weight mg of the body supported on it.

6·13. Illustrative Examples.

Ex. 1. *A train, whose mass is 300 tons, moves at the rate of 60 miles per hour; after steam is shut off, it is brought to rest by the brakes in 50 yds. Find the force exerted, assuming it to be uniform.*

[*C. U. 1934*]

60 miles per hour = 88 ft. per sec. is the velocity when steam is shut off and brakes applied.

Velocity becomes zero, after a distance 50 yds. = 150 feet is described.

Hence, f denoting the acceleration in ft./sec² produced by the brakes opposing motion,

$$0 = 88^2 - 2.150f,$$

$$\text{or,} \quad f = \frac{88^2}{300} \text{ ft./sec}^2.$$

The mass of the train being 300 tons = 300 × 2240 lbs., the force of retardation exerted on the train due to the application of the brakes is thus

$$300 \times 2240 \times \frac{88^2}{300} \text{ poundals}$$

$$= 300 \times 2240 \times \frac{88 \times 88}{300} \times \frac{1}{32} \text{ lbs. wt.}$$

$$= 2240 \times 88 \times 88 \times \frac{1}{32} \times \frac{1}{2240} \text{ tons wt.}$$

$$= 242 \text{ tons wt.}$$

N. B. Note that in applying the formula $P = mf$ here, m and f are expressed in lbs. and ft/sec units, and the result is obtained thereby in poundals, which, on dividing by 32, is reduced to lbs. wt.

Ex. 2. *A particle of mass 20 lbs. falls from a height of 25 feet and penetrates into the ground. If the resistance to penetration is constant and equal to a force of 1020 lbs. weight, find the distance through which it penetrates.*

Just before penetration into the ground, the velocity of the particle is that due to a free fall from a height 25 feet, and is given by

$$v^2=2.g.25, \quad \text{or,} \quad v=\sqrt{2\times 32\times 25}$$
$$=40 \text{ ft./sec.}$$

Here the upward force of resistance to penetration is 1020 lbs. wt. and the weight of the body which is a force acting downwards, is 20 lbs. wt. Thus, the resultant upward force on the body = 1000 lbs. wt. = 32000 poundals.

Therefore the acceleration opposite to the direction of motion is

$$\frac{32000}{20} \text{ ft. sec. units} = 1600 \text{ ft/sec}^2.$$

Hence, x ft. being the depth penetrated, when the velocity of the particle is zero,

$$0=40^2-2\times 1600\times x,$$

$$\text{or,} \quad x=\frac{40\times 40}{2\times 1600}=\tfrac{1}{2} \text{ ft.}=6 \text{ inches.}$$

Ex. 3. *A train travelling on a level road at the rate of* 15 *miles per hour comes to the foot of an incline of* 1 *in* 160 *and steam is then turned off. How far will the train go up the incline before it comes to rest, if the resistance due to friction etc. be* 14 *lbs. wt. per ton ?*

'*An incline of 1 in 160*' means that the plane is inclined at an angle α to the horizon where $\sin\alpha=\frac{1}{160}$.

Let m lbs. be the mass of the train.

Since the train is moving up the incline, the component of its weight down the plane, *i.e.*, $mg \sin\alpha$, is a retarding force, and the force of resistance due to friction etc. $=14 \dfrac{m}{2240}\cdot g$ poundals.

$$\left[\because \quad m \text{ lbs.}=\frac{m}{2240} \text{ tons.}\right]$$

Hence, the total retarding force along the plane

$$=m\left\{32\cdot\frac{1}{160}+\frac{14\times 32}{2240}\right\} \text{ poundals.}$$

$\therefore$ retardation $=32\times\frac{1}{160}+\frac{14\times32}{2240}=\frac{1}{5}+\frac{1}{5}=\frac{2}{5}$ ft. per sec^2.

$$\left[f=\frac{P}{m}\right]$$

15 miles per hour $=22$ ft. per sec.

$\therefore$ if x be the distance traversed by the train up the plane before coming to rest,

then, $\quad 0=22^2-2.\frac{2}{5}.x.$

$$\therefore \quad x=\frac{22\times22}{2\times\frac{2}{5}}=605 \text{ ft.}$$

6·14. Analytical equations of motion.

It is given in Newton's Second Law of Motion that if a force P acting on a particle of mass m in a given direction produces acceleration f (which is in the same direction) then

$$P=mf.$$

From this we have analytically the following equations of motion of a particle *in a given direction,*

$$P=m\frac{d^2x}{dt^2}=m\frac{dv}{dt}=mv\frac{dv}{dx}$$

x being the displacement of the particle from a fixed origin in the given direction at a time t and v the velocity at that instant.

6·15. Illustrative Examples.

Ex. 1. *A particle of mass m moves towards the centre of attraction, starting from rest at a distance 'a' from the centre. If its velocity when at any distance x from the centre varies as $\sqrt{\frac{a^3-x^3}{x^3}}$, show that the law of force varies inversely as the fourth power of the distance.*

Here, $\quad v=\sqrt{\frac{a^3-x^3}{x^3}}.$

$$\therefore \quad v^2=\frac{a^3-x^3}{x^3}=\frac{a^3}{x^3}-1.$$

$$\frac{dv}{dx} = -\frac{3}{2}\cdot\frac{a^3}{x^4}.$$

$\therefore$ the law of force $= m.v\dfrac{dv}{dx} = -\dfrac{3ma^3}{2}\cdot\dfrac{1}{x^4}.$

Hence, the law of force varies inversely as x^4, the fourth power of the distance.

Ex. 2. *A particle of mass m is always acted on by a force towards a fixed point, whose magnitude at a distance x from the point is* $\dfrac{m\mu}{x^3}$. *If it starts from rest at a distance c from the point, find the velocity at a distance* $\frac{1}{2}c$.

Here, the equation of motion is

$$m\frac{d^2x}{dt^2} = -\frac{m\mu}{x^3},$$

or,
$$\frac{d^2x}{dt^2} = -\frac{\mu}{x^3}.$$

Multiplying by $2\dfrac{dx}{dt}$ and integrating with respect to t,

$$\left(\frac{dx}{dt}\right)^2 = \frac{\mu}{x^2} + D.$$

But initially $x = c$, $\dfrac{dx}{dt} = 0$, $\quad\therefore\quad D = -\dfrac{\mu}{c^2}$,

$$\therefore \quad \left(\frac{dx}{dt}\right)^2 = \mu\left(\frac{1}{x^2} - \frac{1}{c^2}\right).$$

When $x = \frac{1}{2}c$, the velocity v is given by

$$v^2 = \mu\left(\frac{4}{c^2} - \frac{1}{c^2}\right) = \frac{3\mu}{c^2},$$

$$\therefore \quad v = \frac{\sqrt{3\mu}}{c}.$$

Examples on Chapter VI

1. A constant force acts upon a mass of 8 lbs. during 4 seconds from rest and then ceases; in the next 4 seconds it is found that the mass describes 64 feet. Find the magnitude of the force.

2. A body acted upon by a uniform force moves through 1 metre in 10 seconds from rest. Find the ratio of the force to the weight of the body.

3. A mass m lbs. is acted on by a constant force of P poundals, under which, in t seconds, it moves a distance x ft. from rest, and acquires a velocity of v ft. per second. Show that

$$x = \frac{1}{2} \cdot \frac{mv^2}{P}.$$

4. A force equal to the wt. of 1000 grammes acts on a mass of 200 grammes for half a minute. Find the velocity acquired by the mass. [*C. U. 1932*]

5. A heavy body weighing 128 lbs. is being raised from the bottom of a pit 100 ft. deep with a uniform force of 160 lbs. wt. Find the time taken by the body to reach the top of the pit.

6. Each of two bodies at rest on a smooth horizontal table attracts the other with the same force irrespective of the distance between the two bodies. Show that in any time they move over spaces which are inversely as their masses. If the masses of the two bodies be 9 and 16 lbs., the constant force be 2 lbs. wt. and the distance between the masses be 32 ft., find after what time they would meet.

7. A mass of 4 lbs. falls 200 ft. from rest and is then brought to rest by penetrating 2 feet into some mud. Find the average thrust of the mud on it. [*U. P. 1940*]

8. A railway train whose mass is 100 tons, moving at the rate of 60 miles per hour in a straight line is brought to rest in 10 seconds by the application of a uniform force. Find how far the train moves during the time for which the force is applied, and calculate the magnitude of the force. [*C. U. 1940*]

9. A body of mass 6 lbs. has been falling from rest under the action of gravity for 4 seconds ; find what vertical force applied to it will bring it to rest in 64 ft.

10. A ball of mass 100 gms. falls freely through a distance of 10 metres from rest. It is then brought to rest by a uniform force acting vertically upwards on it in 1 sec. A second force similarly acting on it would stop it in 2 secs. Show that the first force is $1\frac{5}{12}$ times the second.

[*Assume* $g = 980$ *cms./sec*^{2}.]

11. A bullet weighing half-an-ounce leaves the muzzle of a rifle-barrel 2 ft. long, with a velocity of 2000 ft. per sec. Find the force acting on the bullet in the barrel, assuming it to be uniform ; and also the time taken by the bullet to traverse the barrel. [*C. U. 1938*]

12. A shot of mass 100 lbs. moving at the rate of 1600 ft. per sec. strikes a fixed target. How far will the shot penetrate the target, assuming that it offers an average resistance of the weight of 12000 tons ? [*C. U. 1933*]

13. Find the velocity of a 4 lbs. shot that will just penetrate through a wall 10 inches thick, the resistance being 42 tons wt. [*U. P. 1936*]

14. A train running at 15 miles per hour comes to the foot of an incline of 1 in 280. The resistance due to friction etc. is 8 lbs. wt. per ton. How far will the train go up the incline before stopping ?

15. A railway train exclusive of engine weighs 435 tons. and starting along a level line from rest attains a speed of 40 miles per hour in 7 minutes. Calculate the average pull between the engine and the train, taking the resistance to be 15 lbs. wt. per ton. [*C. U. 1935*]

16. A train runs from rest for 1 mile down an incline of 1 in 100. If the resistance be equal to 8 lbs. wt. per ton, how far will the train be carried along the horizontal level at the foot of the incline ? [*U. P. 1941*]

17. A train is moving on a horizontal railroad. Assuming the weight of the train (exclusive of the engine) to be 160 tons and the resistance arising from friction etc. to

be 8 lbs. wt. per ton, find the pull between the engine and the train (i) when the velocity of the train is uniform, and (ii) when it is moving with an acceleration of 4 ft./sec^2.

18. A thief jumps off the terrace of a building with a heavy suitcase on his head, and falls vertically. What would be the pressure of the suitcase on his head while he is falling ?

19. A boy with a basket of 8 pounds of sweets hanging from his hand is descending in a lift. The lift starts down with an acceleration of 2 ft. per sec^2, reaching a steady speed which it keeps up till it slows down at the rate of 4 ft. per sec^2. Find in pounds weight the pressure on the hand of the boy during the *three stages* of the lift's descent.

✓ **20.** A man weighing 12 stones is descending in a lift with acceleration 8 ft./sec^2. Find the thrust of his feet on the lift. Calculate the same when he is ascending with the same acceleration. What would happen to this thrust if the chain of the lift broke (i) during descent, (ii) during ascent ? [*C. U. 1943*]

21. A thin glass plate can just support a weight of 27 lbs. A body is placed on it and the plate is raised with the body on it with a gradually increasing acceleration. It is found that the plate breaks when the acceleration is 4 ft./sec^2. Find the mass of the body.

22. A man is raising by means of a rope, 28 lbs., of water in a bucket weighing 7 lbs., and he feels a uniform pressure of 42 lbs. wt. on his hand. If the depth of the well be 80 feet, find the time he takes to raise the water to the surface. Find also the pressure on the bottom of the bucket.

23. A body weighs 2 lbs. at the equator as seen by a spring-balance, and is observed to weigh $\frac{1}{8}$ of an oz. more at Calcutta, by using the same spring-balance. A boy at Calcutta can throw a ball 16 ft. vertically upwards. How high can he send the same ball at the equator ?

24. A load W is raised by a rope, from rest to rest through a height h; the greatest tension which the rope can safely bear is nW. Show that the least time in which the ascent can be made is $\left\{\frac{2nh}{(n-1)g}\right\}^{\frac{1}{2}}$.

25. A particle of mass m starts from rest at a distance a from the origin. If the force acting at any instant is $ma \cosh t$, show that its distance from the origin at time t is $a \cosh t$ and its velocity at time t is $a \sinh t$.

26. A particle of mass m moves in a straight line under a repulsive force mn^2x away from a fixed point O on the line, x being the distance of the particle from O and n being a constant. If it be initially at a distance a and moving with the velocity nu, show that at time t it will be at a distance

$$a \cosh nt + u \sinh nt.$$

27. A particle, whose mass is m, is acted upon by a force $m(x + a^4x^{-3})$ towards the origin; if it starts from rest at a distance a, show that it will arrive at the origin in time $\frac{\pi}{4}$.

28. A particle moves in a straight line under the action of a repulsive force proportional to its distance from a point O on the line. If the particle be initially projected with a velocity V *towards* O when it is at a distance a from this point, find the velocity of the particle and its distance from O after time t.

29. A point moves in a straight line towards a centre of force under the action of a force which varies inversely as the cube of the distance, starting from rest at a distance a from the centre of force; find the time of reaching a point distant b from the centre of force and the velocity there.

30. A particle of mass m, moving along the x-axis, is acted on by an attractive force which is given by the formula $\frac{2mk^2a^2}{x^2}$ for $x > a$, and by the formula $\frac{2mk^2x}{a^2}$ for

$x < a$. If the particle starts from rest at a distance $2a$, prove that it will reach the origin with velocity $2k\sqrt{a}$.

31. A particle of mass m moves in a straight line towards a centre of force which attracts at a distance x with a force $\frac{mn^2}{x^2}$, starting from rest at a distance 8 ft. from the centre. Show that the time required to reach the centre of force is $\frac{8\pi}{n}$.

32. If in the above example, t_1 be the time of motion from the distance 8 ft. to the distance 4 ft. and t_2 be the time of motion from the distance 4 ft. to the origin, show that

$$\frac{t_1}{t_2} = \frac{\pi + 2}{\pi - 2}.$$

ANSWERS

1. 1 lb. wt. **2.** 2 : 981. **4.** 147150 cms./sec.

5. 5 secs. **6.** $2\frac{3}{5}$ secs. **7.** 404 lbs. wt.

8. 440 ft. ; $27\frac{1}{2}$ tons wt. **9.** 30 lbs. wt.

11. 31250 poundals ; ·002 secs. **12.** $1\frac{1}{4}$ inches.

13. 1120 ft./sec. **14.** $1058\frac{2}{3}$ feet.

15. $10778\frac{1}{3}$ lbs. wt. **16.** $1\frac{4}{5}$ miles.

17. 1280 lbs. wt. ; $20\frac{4}{7}$ tons wt. **18.** 0.

19. $7\frac{1}{2}$ lbs. wt., 8 lbs. wt., 9 lbs. wt.

20. 9 stone wt. ; 15 stone wt. ; the thrust becomes zero in either case.

21. 24 lbs. **22.** 5 secs. ; $33\frac{3}{8}$ lb. wt. **23.** 16·1 ft.

28. $a\sqrt{\mu}\sinh\sqrt{\mu}t - V\cosh\sqrt{\mu}t$,

$a\cosh\sqrt{\mu}t - \left(\frac{V}{\mu}\right)\sinh\sqrt{\mu}t$.

29. If the law of force $= \frac{\mu}{x^3}$, then time required

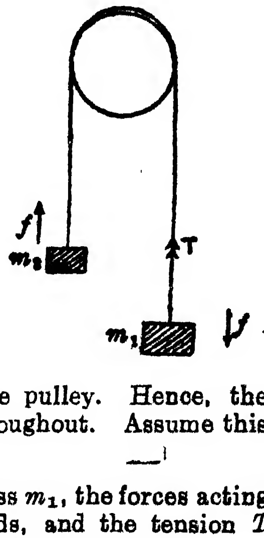
$$= \frac{a\sqrt{a^2 - b^2}}{\sqrt{\mu}}, \quad \text{vel.} = \frac{\sqrt{\mu}}{ab}\sqrt{a^2 - b^2}.$$

CHAPTER VII

MOTION OF CONNECTED SYSTEMS

7.1. *Two particles of masses m_1 and m_2 $(m_1 > m_2)$ are connected by a* **light inextensible** *string passing over a* **light smooth** *pulley, and are allowed to hang freely. To find the resulting motion, and the tension of the string.*

Let f be the acceleration with which m_1 descends. As the *string is inextensible*, the displacement of m_1 downwards will always be equal to that of m_2 upwards, and hence at every instant the velocity of m_2 upwards will be equal to that of m_1 downwards. Accordingly, the acceleration (rate of change of velocity) of m_2 upwards will also be the same as that of m_1 downwards, namely f.



As the *string is light*, the tension of the string on each side of the pulley will be the same throughout its length, and as the *pulley is light and smooth*, the tension does not change along the string as it passes from one side to the other over the pulley. Hence, the tension of the string is constant throughout. Assume this tension to be T in absolute units.

Confining our attention to the mass m_1, the forces acting on it are its weight $m_1 g$ downwards, and the tension T upwards, and the acceleration being f downwards,

$$m_1 g - T = m_1 f. \qquad \cdots \quad \text{(i)}$$

Similarly, considering the upward motion of m_2,

$$T - m_2 g = m_2 f. \qquad \cdots \quad \text{(ii)}$$

Adding (i) and (ii), and dividing by $m_1 + m_2$,

$$\mathbf{f = \frac{m_1 - m_2}{m_1 + m_2}\, g.}$$

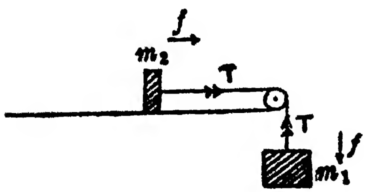
Then from (ii), $\mathbf{T} = m_2 g + \dfrac{m_2(m_1 - m_2)}{m_1 + m_2} g = \dfrac{\mathbf{2m_1m_2}}{\mathbf{m_1 + m_2}} \mathbf{g}.$

Note 1. If $m_1 < m_2$, the downward acceleration f of m_1 is negative, *i.e.*, m_1 has a positive upward acceleration $(m_2 - m_1)g/(m_2 + m_1)$ and m_2 has the same acceleration downwards.

Note 2. As the string presses the pulley downwards on both sides, the pressure on the pulley (being the resultant of two equal and parallel tensions T, T) $= 2\mathrm{T} = \dfrac{4m_1m_2}{m_1 + m_2} \cdot g$.

7·2. *Two particle of masses m_1 and m_2 are connected by a light inextensible string passing over a light smooth pulley at the edge of a smooth horizontal table, m_2 lying on the table and m_1 hanging vertically. To determine the resulting motion and the tension of the string.*

Let f be the acceleration with which m_1 descends. As the string is inextensible, m_2 will move on the table along the string with the same acceleration.



As the string is light, and the pulley light and smooth, the tension of the string will be constant throughout its length. Let T be this tension.

Considering the motion of m_1 downwards,

$$m_1 g - T = m_1 f. \qquad \cdots \quad \text{(i)}$$

Considering the horizontal motion of m_2 on the table,

$$T = m_2 f. \qquad \cdots \quad \text{(ii)}$$

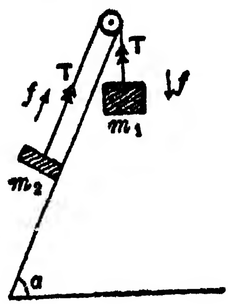
From (i) and (ii), adding and dividing by $m_1 + m_2$, we get

$$\mathbf{f} = \frac{\mathbf{m_1}}{\mathbf{m_1 + m_2}} \mathbf{g}, \text{ and then from (ii), } \mathbf{T} = \frac{\mathbf{m_1 m_2}}{\mathbf{m_1 + m_2}} \mathbf{g}.$$

Note. The *pressure on the pulley* here, being the resultant of two equal tensions T, T of the two parts of the string pressing it in perpendicular directions, is $2T \cos 45° = T\sqrt{2}$.

7·3. *Two particles of masses m_1 and m_2 are connected by a light inextensible string passing over a light smooth pulley placed at the top of a smooth inclined plane of inclination α to the horizon, m_1 hanging freely and m_2 resting on the inclined plane, the portion of the string on the inclined plane being parallel to the line of greatest slope. When the system is allowed to itself, to find the resulting motion.*

Let f be the acceleration with which m_1 descends. As the string is inextensible, m_2 will rise up the plane with the same acceleration.



Let T be the tension of the string, which, since the string is light and the pulley light and smooth, must be constant throughout the string.

Considering the motion of m_1 vertically downwards,

$$m_1 g - T = m_1 f. \qquad \cdots \quad \text{(i)}$$

Again, considering the motion of m_2 up the plane, and remembering that the component of its weight along the plane is $m_2 g \sin \alpha$ downwards,

$$T - m_2 g \sin \alpha = m_2 f. \qquad \cdots \quad \text{(ii)}$$

From (i) and (ii), solving, we get

$$\mathbf{f} = \frac{\mathbf{m_1 - m_2 \sin \alpha}}{\mathbf{m_1 + m_2}}\,\mathbf{g}, \quad \boxed{\mathbf{T} = \frac{\mathbf{m_1 m_2 (1 + \sin \alpha)}}{\mathbf{m_1 + m_2}}\,\mathbf{g}}$$

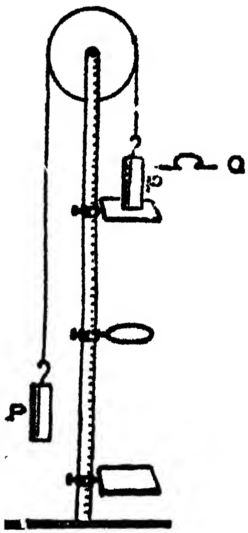
Note 1. If $m_1 < m_2 \sin \alpha$, f is negative, so that m_1 will rise upwards with positive acceleration $\dfrac{m_2 \sin \alpha - m_1}{m_1 + m_2} g$, and m_2 will descend down the plane with the same acceleration.

Putting $\alpha = \frac{1}{2}\pi$ and zero respectively, we get the result of Arts. 7·1 and 7·2 as particular cases of the above general result.

Note 2. *The pressure on the pulley* here is easily seen to be $2T \cos \frac{1}{2} (90° - \alpha)$, since the two portions of the string pressing the pulley on the two sides are inclined at an angle $(90° - \alpha)$.

7·4. Atwood's Machine ; Verification of the Laws of motion.

Atwood's Machine (constructed on the principle of Art. 7·1) is generally used to verify Newton's Laws of motion, as also for a rough determination of the value of g at any place.



It consists essentially of a graduated vertical stand, at the top of which a very light smooth pulley is attached. Over this passes a fine silk thread, at the two extremities of which two equal cylindrical brass weights P, P are attached. There are two platforms and a ring which can be clamped by screws at any desired points of the stand, the ring being somewhere between the platforms. There is another piece of small weight Q, called a rider, with projected arms of a shape shown in the figure, which can be placed horizontally over a cylindrical weight P, and the ring is of such diameter that it allows the cylindrical weight P to pass through it easily, but arrests the rider. Initially the rider is placed on one weight P which rests on the upper platform near about the top of the stand. The upper platform can be instantaneously dropped,* when the system, with a weight $P+Q$ on one side and P on the other, begins to move. After some time the weight P having the rider on it just passes through the ring, when the rider is arrested and the subsequent motion of the system is with equal weights P, P on either side. Finally the motion comes to an end when the weight P passing through the ring reaches the lower platform.

*Sometimes the upper platform is avoided by using a spring catch holding the lower weight P. When starting motion, this catch is released.

The distances moved through by the system during the two stages of motion, first from start till the rider is arrested, and the second from this instant till the end, are noted on the graduated stand, and the corresponding times taken are recorded by stop-watches. Let the distances be h_1 and h_2 and the times be t_1 and t_2. Clearly h_1 is the distance from the top of the cylindrical weight P initially resting on the upper platform to the ring whereas h_2 is the distance from the ring to the top of P when the latter meets the lower platform.

Now assuming the truth of Newton's Second Law of motion, the formula $P=mf$ has been deduced, and thence, as in Art. 7·1, we get the acceleration during the first stage of motion given by

$$f=\frac{(P+Q)-P}{(P+Q)+P}\cdot g=\frac{Q}{2P+Q}\,g. \qquad \cdots \quad \text{(i)}$$

With this acceleration, h_1 is the distance travelled in time t_1 with starting velocity zero.

Hence, $$h_1=\frac{1}{2}\cdot\frac{Q}{2P+Q}g\,.\,t_1^2. \qquad \cdots \qquad \cdots \quad \text{(ii)}$$

With P and Q known, and h_1 and t_1 noted, the value of g is found.

Altering h_1 at pleasure by altering the point of fixation of the ring and noting t_1 in each case, we shall get practically the same value of g.

Conversely, assuming g to be known, the observed values of h_1 and t_1 will be seen to satisfy (ii) in all cases, verifying the correctness of the calculated value of the acceleration f as in (i), and thereby indirectly verifying the truth of the assumption of Newton's Second Law of motion.

Again, at the end of the first part of the motion, the velocity acquired by the system is given by

$$v^2=2\cdot\frac{Q}{2P+Q}\,g.h_1. \qquad \cdots \qquad \cdots \quad \text{(iii)}$$

It will be observed that this value of v exactly equals $\frac{h_2}{t_2}$, *i.e.*, $h_2=t_2v$. By altering h_2 by shifting the lower

platform, and noting t_2 in each case, the same result will be found in every case to hold. This shows that during the second stage of motion of the system, when the weights on the two sides are equal, the velocity of the system once acquired is uniform in absence of any resultant force on the system, thus giving an indirect verification of the first law.

In calculating the acceleration in Art. 7·1, which gives (i) in the present case, the tension was considered constant throught the string, and this involved the assumption of the third law of motion. The experimental verification of the calculated value of f above thus adduces an evidence as to the truth of the third law as well.

7·5. Illustrative Examples.

* Ex. 1. *A mass of 3 lbs. descending vertically, draws up a mass of 2 lbs. by means of a light string passing over a pulley; at the end of 5 seconds the string breaks; find how much higher the 2 lbs. mass will go.* [*U. P. 1934 ; Pat. 1935*]

The acceleration of the connected system in this case is

$$\frac{3-2}{3+2}.g = \frac{32}{5} \text{ ft./sec}^2.$$

Hence, after 5 seconds from start, the velocity of the system is

$$\frac{32}{5} \times 5 = 32 \text{ ft./sec.}$$

which thus represents the upward velocity of the 2 lbs. mass.

The string now breaking, the 2 lbs. mass is now free, and has a downward acceleration $g=32$ ft./sec^2 due to gravity.

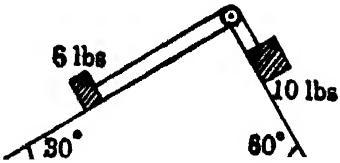
Hence the further height it rises from this instant before its upward velocity becomes zero being x ft.,

$$0^2 = 32^2 - 2.32.x,$$

or, $$x = 16 \text{ feet.}$$

Ex. 2. *Two smooth inclined planes of equal heights, whose inclinations to the horizon are 30° and 60°, are placed back to back; two bodies of masses 6 and 10 lbs. placed on them respectively, are connected by a light, inextensible string, passing over a smooth pulley at the common vertex of the planes. Find the tension in the string and the acceleration of the system.* [*U. P. 1937*]

Let f be the common acceleration of the system with which the 10 lbs. mass descends down the second plane, or the 6 lbs. mass ascends up the first plane. Also let T be the tension in the string.



Considering the motion of the masses along the respective planes, we get,

$$T-6g\sin 30^\circ=6f$$
$$10g\sin 60^\circ-T=10f.$$

From these, adding,

$$10g\cdot\frac{\sqrt{3}}{2}-6g\cdot\frac{1}{2}=16f,$$

or, $$f=(5\sqrt{3}-3)\frac{g}{16}=2(5\sqrt{3}-3)\text{ ft./sec}^2.$$

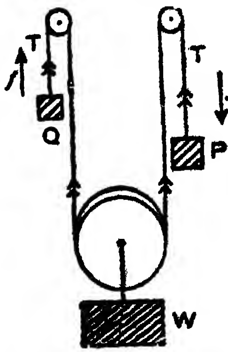
Again, putting this value of f in one of the equations,

$$T=6\times32\times\tfrac{1}{2}+6\times2(5\sqrt{3}-3)$$
$$=60(\sqrt{3}+1)\text{ poundals.}$$

Ex. 5. *A pulley carrying a total load W hangs in a loop of a cord which passes over two fixed pulleys, and has unequal weights P and Q freely suspended from its ends, each segment of the cord being vertical. Shew that W will remain at rest provided*

$$\frac{1}{P}+\frac{1}{Q}=\frac{4}{W}.$$ [*C. U. 1939, '42*]

Let f be the acceleration with which P descends. As W remains at rest, the string slipping under it, Q will ascend with the same acceleration. Let T be the tension of the string, which is evidently the same throughout its length.



Then, considering the motions of P and Q,

$$P-T=\frac{P}{g}\cdot f$$
$$T-Q=\frac{Q}{g}f.$$

[P/g and Q/g being the masses of the weights P and Q]

From these, eliminating f, we ultimately get

$$T=\frac{2PQ}{P+Q}.$$

Now, since W remains at rest, its weight is supported by the upward tensions of the two parts of the string on two sides of it, and thus

$$W=2T=\frac{4PQ}{P+Q}; \quad \therefore \quad \frac{4}{W}=\frac{P+Q}{PQ}=\frac{1}{P}+\frac{1}{Q}.$$

Examples on Chapter VII

[*In the following examples, unless otherwise stated, the strings are to be considered as weightless and inextensible, and the pulleys as smooth and of negligible mass.*]

1. Two masses 3 and 7 lbs. are connected by a light string passing over a smooth pulley, and hang freely. Motion is allowed to ensue from rest. In what time will the heavier mass descend 10 feet? Find also the distance described during the next $1\frac{1}{2}$ seconds.

2. Two scale-pans of mass 100 gms. each, hang freely from the two ends of a string passing over a smooth pulley. On these are placed two weights, 398 and 383 gms. respectively, and the system is allowed to start with the pans in the same horizontal level. Find the velocity of the system when the distance between the pans is 60 cms.

3. Two scale-pans, each of mass 2 ozs., are suspended by a light string over a smooth pulley; a mass of 14 ozs. is placed on one and 6 ozs. on the other. Find the tension of the string, and the pressures on the scale-pans.

4. If two masses each equal to 6 lbs., connected by a string hang over a pulley, and a mass of 4 lbs, be added to one of them, find by how much the pressure on the pulley is increased

5. Two unequal masses connected by a string hang over a pulley. Show that the pressure on the pulley is less than the sum of the weights.

6. Two unequal masses connected by a string hang over a pulley ; if the sum of the masses be constant, shew that the greater the acceleration, the less is the tension in the string. [*C. U. 1935*]

7. If two unequal weights be contained in scale-pans connected by a string passing over a smooth pulley, prove that if the weights of the pans be negligible, the pressure between each pan and the contained whight is equal to the tension of the string.

8. A flexible heavy chain of length $2l$, is moving over a smooth fixed pulley, the two unequal portions of it hanging vertically. Prove that at the instant when its middle point is at a distance x below the pulley, the acceleration with which it is moving is $\frac{x}{l}g$.

9. A stone of mass 1 kilogram breaks into two pieces. These are placed, one on each of two equal scale-pans of mass 45 gms., suspended from the two extremities of a string passing over a smooth pulley, and it is observed that the system moves through 10 cms. in $\frac{1}{2}$ sec. Find the mass of the heavier piece.

10. Two bodies of masses 14 lbs. and 18 lbs. connected by a long string hang over a pulley, and motion is allowed to start from rest. After 3 secs. the string is cut. What time after this will the lighter body come to its starting position ?

11. A weight of 300 lbs. is to be raised through a certain height by a cord passing over a fixed smooth pully. It is found that a constant force P pulling the cord at its other end for three-fourths of its ascent communicates sufficient velocity to enable it to reach the required height. Find P.

12. Two masses P and Q $(P > Q)$ connected by a string hang over a pulley. After 1 sec. from start, P is suddenly stopped, and instantly let go. Find the time that elapses before the string becomes tight again.

13. Two equal masses hang at rest over a smooth pulley; one is projected upwards with a velocity of 96 ft. per sec.; in what time will the string become tight again?

14. A mass of 10 lbs. descending vertically, draws up a lighter mass, by means of a thin string passing over a smooth pulley; at the end of 2 secs. the string breaks; if the lighter body rises 4 ft. higher, find its mass.

15. A mass of 9 lbs. is attached to one end of a string and masses of 7 and 4 lbs. to the other end, and the whole is hung up over a pulley. The system is allowed to move for 15 secs., when the 4 lbs. weight is cut away. How long will it be before the system comes instantaneously to rest? [*U. P. 1939*]

16. Two light inextensible strings pass over a small smooth pulley. On one side they are attached to masses 3 and 4 lbs. respectively, and on the other to one of 5 lbs. Find the acceleration of the system, and the tensions of the strings.

17. A light string carrying two unequal weights and passing over a smooth pulley can only just bear a tension equal to $\frac{12}{25}$ of the sum of the weights; prove that the least acceleration possible of the system is $\frac{1}{5}g$, and that the lighter mass cannot exceed $\frac{2}{5}$ of the total mass.

18. A mass of 12 lbs. lying on a smooth horizontal table 9 ft. from the edge is drawn along the table by a mass of 4 lbs. hanging freely by means of a light inextensible string passing over a smooth pulley at the edge. How long does it take to reach the edge, and what is the pressure on the pulley?

19. Two masses 5 lbs. and 3 lbs. are connected by a light string passing over a smooth table $2\frac{1}{2}$ ft. wide, at right angles to its edges, the smaller mass starting at a point 4 ft. below the edge. Find the time taken by the larger mass to fall through 6 ft., supposing the smaller mass to pass on to the table without loss of velocity.

20. A light string passing across a smooth table at right angles to two opposite edges has attached to it at the two

ends masses m_1, m_2 $(m_1 > m_2)$ which hang vertically. A particle of mass m is attached to the portion of the string lying on the table. Show that the acceleration of the system when left to itself is

$$\frac{m_1 - m_2}{m_1 + m_2 + m} g.$$

21. A smooth inclined plane whose height is one-half of its length has a small smooth pulley at the top, over which a string passes. To one end of the string is attached a mass of 22 lbs. which rests on the plane, while from the other end which hangs vertically, is suspended a mass of 14 lbs., and the masses are free to move. Find the acceleration, and the distance traversed by either mass in 2 seconds. Find also the pressure on the pulley. [*C. U. 1941*]

22. Masses 6 and 2 lbs. rest on two inclined planes, each of elevation 30°, and are connected by a string passing over the common vertex ; find the acceleration, and the tension of the string.

23. Two bodies P and Q having masses 9 and 6 lbs. respectively are connected by a string passing over the top of an inclined plane of inclination 30° to the horizon. One body rests on the plane and the other hangs vertically. Show that P hanging vertically will drag Q up the whole length of the plane in half the time that Q hanging vertically will take to drag P up the plane.

24. A mass M is drawn up a smooth inclined plane of height h and length l by means of a string passing over the vertex of the plane, from the other end of which hangs a mass m. Show that in order that M may just reach the top of the plane, m must be detached after M has moved through a distance

$$\frac{M+m}{m} \cdot \frac{hl}{h+l}.$$

25. Two weights W and W' are connected by a light string passing over a smooth pulley. If the pulley moves

vertically upwards with an acceleration equal to that of gravity, show that the tension of the string is

$$\frac{4WW'}{W+W'}.$$ [C. U. 1944]

26. If in the above case the acceleration of the pulley vertically upwards be f, find the acceleration of each weight and the tension of the string.

27. A light rope hangs over a smooth pulley. A monkey of weight 5 stones climbs down the portion of the rope on one side with an acceleration of 2 ft./sec^2. Find with what acceleration another monkey of weight 4 stones will climb up the portion of the rope on the other side, so that the rope may remain at rest.

28. On one side of a light string passing over a smooth pulley, a weight W hangs. A boy takes hold of the other end, and begins to climb up the rope with a uniform acceleration, rising 16 ft. in 2 secs. If W remains at rest, show that the weight of the boy is $\frac{4}{5}W$.

29. One end of a string is fixed ; it then passes under a movable pulley to which a weight W is attached. The string then passes over a fixed pulley, and a weight P is attached to its other end, all the three sections of the string being vertical, Show that neglecting the masses of the pulleys, the acceleration with which W ascends is

$$\frac{2P-W}{W+4P}g.$$

Find also the tension of the string. [C. U. 1937]

30. A small pulley carrying a total load W hangs in a loop of a cord which passes over two fixed pulleys, and has unequal weights P and Q freely suspended from its ends, each segment of the cord being vertical. Show that W will ascend with acceleration

$$\frac{4PQ-W(P+Q)}{4PQ+W(P+Q)}g.$$

31. Two particles, of masses m_1 and m_2, lie together on a smooth horizontal table. A string which joins them hangs over the edge in the form of a loop, and supports a smooth heavy pulley of mass M; show that the pulley descends with an acceleration

$$\frac{M(m_1+m_2)}{4m_1m_2+M(m_1+m_2)}\,g.$$

32. A light string passes over a light fixed pulley; it carries a mass P at one extremity and a light pulley at the other. Another light string passes over the second pulley carrying masses R and Q at its extremities. The system starts from rest. If R always remains at rest, prove that

$$\frac{4}{P}+\frac{1}{Q}=\frac{3}{R}. \qquad \text{[C. H. 1964, '65]}$$

ANSWERS

1. $1\frac{1}{4}$ seconds ; 30 ft. **2.** 30 cms./sec.

3. $\frac{2}{3}$ lbs. wt. ; $\frac{7}{12}$ lbs. wt. ; $\frac{1}{2}$ lbs. wt. **4.** 3 lbs. wt.

9. 600 gms. **10.** $1\frac{1}{2}$ seconds. **11.** 400 lbs. wt.

12. $\frac{P-Q}{P+Q}$ seconds. **13.** 3 seconds. **14.** 6 lbs.

15. 12 seconds. **16.** $5\frac{1}{3}$ ft./sec^2 ; $2\frac{1}{2}$ and $3\frac{1}{3}$ lbs. wt.

18. $1\frac{1}{2}$ seconds ; $3\sqrt{2}$ lbs. wt. **19.** $1\frac{1}{5}$ seconds.

21. $2\frac{2}{3}$ ft./sec^2 ; $5\frac{1}{3}$ ft. ; $\frac{77}{8}\sqrt{3}$ lbs. wt.

22. 8 ft./sec^2 ; $1\frac{1}{2}$ lbs. wt.

26. $f_1=\frac{2W'}{W+W'}(g+f)-g$ and $f_2=\frac{2W}{W+W'}(g+f)-g$, both measured upwards ; $T=\frac{2WW'}{W+W'}\left(1+\frac{f}{g}\right)$. **27.** $5\frac{1}{3}$ ft./sec^2. **29.** $\frac{3WP}{W+4P}$.

CHAPTER VIII

PROJECTILES

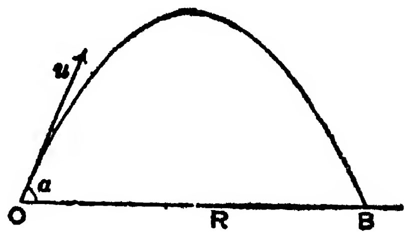
8·1. In Chapter V we have considered rectilinear motion only under gravity, as for instance, when a particle is projected vertically upwards, or when aparticle moves on an inclined plane, being projected along the line of greatest slope. In this chapter we shall consider free motion under gravity, when a particle is projected in any direction in space. We shall however, in order to confine ourselves to a simpler case, neglect the resistance of air and consider the motion to be in vacuo. In this case, on account of the acceleration due to gravity in the vertically downward direction, the vertical component of velocity will continually change, whereas, the component velocity in the horizontal direction will remain unchanged, as there is no acceleration in that direction. As a result, the resultant direction of motion will continually change, and the particle will describe a curved path. It will be shown later (*Art. 8·5*), that this path is a parabola.

A body thrown in any direction in space is defined to be a **projectile,** and the curved path it describes in space is called the **trajectory.** The initial velocity with which it is projected is defined to be the **velocity of projection** and the inclination to the horizon of the direction in which it is initially projected is defined as the **angle of projection.** The distance of the point at which it falls on a plane is defined as the **range** on the plane, and the time interval from start till it meets the plane, *i.e.*, for which it remains in air, is called the **time of flight.**

8·2. Horizontal range and time of flight of a projectile.

Let a particle be projected from O with a velocity u, at an angle α to the horizon. Let R ($= OB$) be the range on the horizontal plane through O, and T the corresponding time of flight.

The initial vertical component of velocity is clearly $u \sin \alpha$, and the acceleration in that direction on account of gravity is $-g$. After time T, when the particle reaches the horizontal plane through O at B, the net vertical displacement is zero, and hence, confining our consideration to the motion of the particle in the vertical direction only,*



$$u \sin \alpha . T - \tfrac{1}{2} g T^2 = 0,$$

giving †, $T = \dfrac{2u \sin \alpha}{g}$

$$= \frac{2}{g} \textit{ (initial vertical velocity)}.$$

The initial horizontal component of velocity is $u \cos \alpha$, and as there is no acceleration in this direction, the above velocity remains unchanged throughout the motion. Hence,

$$OB = \frac{2u \sin \alpha}{g} . u \cos \alpha = \frac{u^2}{g} \sin 2\alpha,$$

$$\textit{i.e.,} \quad R = \frac{u^2}{g} \sin 2\alpha$$

$$= \frac{2}{g} \textit{ (initial vertical velocity} \times \textit{horizontal velocity)}.$$

Cor. 1. Maximum Horizontal Range and Direction for Maximum Range.

From the above value of R, it is apparent that with a given velocity of projection u, the horizontal range is greatest when $\sin 2\alpha$ is greatest, namely unity, which requires $2\alpha = 90°$, or $\alpha = 45°$.

---

*For the justification of considering the motion of the particle in the vertical and horizontal directions separately and independently of one another, see Chapter VI, § 6·9.

†The other solution $T = 0$ corresponds to the starting moment, when also the displacement is zero.

Thus, the *maximum horizontal range* is $\frac{u^2}{g}$, when the *angle of projection is* $45°$,

i.e., the direction of projection for maximum horizontal range bisects the angle between the horizontal and the vertical.

Cor. 2. **For a given horizontal range with a given velocity of projection, there are in general two directions of projection, equally inclined to the direction of maximum range.**

Let u be the velocity of projection of a particle, and let α be the necessary angle of projection in order that the horizontal range may be a given quantity R_1.

Then, $R_1 = \frac{u^2}{g} \sin 2\alpha$. $\therefore \sin 2\alpha = \frac{gR_1}{u^2}$, [a known positive quantity],

$= \sin \theta$ [say, where θ is an acute angle as determined by consulting the Trigonometrical tables].

Then, as α is evidently limited to be within $90°$, 2α is limited to be less than $180°$, and within this limitation, $2\alpha = \theta$, or, $180° - \theta$.

$$\therefore \quad \alpha = \tfrac{1}{2}\theta, \text{ or, } 90° - \tfrac{1}{2}\theta.$$

Thus, there are two possible values of α, and so two directions of projection, giving the same range R_1.

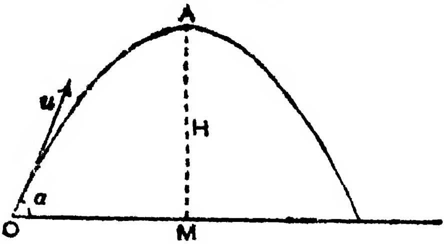
Again, since the angle of projection for maximum range is $45°$, and since $45° - \frac{\theta}{2} = \left(90° - \frac{\theta}{2}\right) - 45°$, the above two directions are equally inclined to the direction of maximum range.

Note. If may be noted that if $R_1 > \frac{u^2}{g}$, $\sin 2\alpha = \frac{gR_1}{u^2}$ becomes greater than unity, and α is impossible, *i.e.*, there is no angle of projection for range greater than $\frac{u^2}{g}$ which is really the maximum range possible.

8·8. Greatest height attained by a projectile, and time to greatest height.

Let a particle be projected from O with a velocity u at an angle α to the horizon. Let T' be the time when it is at the heighest point A of its path, and $H(=AM)$ the greatest height attained.

The initial upward vertical component of velocity is $u \sin \alpha$, and the acceleration in this direction is $-g$ due to gravity. The vertical velocity gradually diminishes, and



at the heighest point of its path A, this vertical velocity becomes zero. The corresponding time being T', and the corresponding vertical displacement being H, we get

$$0 = u \sin \alpha - gT'$$

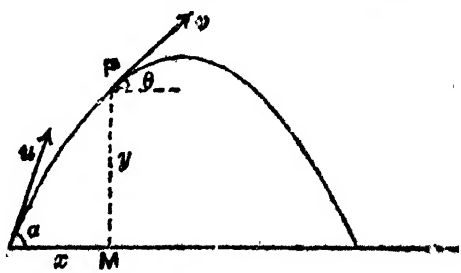
and $$0 = u^2 \sin^2 \alpha - 2gH.$$

$$\therefore \quad T' = \frac{u \sin \alpha}{g} \text{ and } H = \frac{u^2 \sin^2 \alpha}{2g}.$$

Cor. The time to greatest height is half the total time of flight.

84. Position and velocity at any time t.

Let a particle be projected from O with a velocity u at an angle α to the horizon. At any instant t after start, let P be the position of the particle, OM $(=x)$ the horizontal displacement and PM $(=y)$ the vertical displacement. Also, let v be the velocity of the projectile at P, at an angle θ to the horizon.



In the horizontal direction, the initial horizontal component of velocity is $u \cos \alpha$, and this remains unchanged

throughout the motion, as there is no acceleration in this direction.

Hence,

$$v \cos \theta = \text{horizontal component of velocity at } P$$
$$= u \cos \alpha \qquad \cdots \qquad \cdots \qquad \text{(i)}$$

and $$x = OM = \textit{horizontal displacement in time } t$$
$$= u \cos \alpha . t. \qquad \cdots \qquad \cdots \qquad \text{(ii)}$$

Again, initial vertical component of velocity is $u \sin \alpha$, and acceleration in this direction is $-g$ due to gravity; hence,

$$v \sin \theta = \text{vertical component of velocity at } P$$
$$= u \sin \alpha - gt \qquad \cdots \qquad \cdots \qquad \text{(iii)}$$

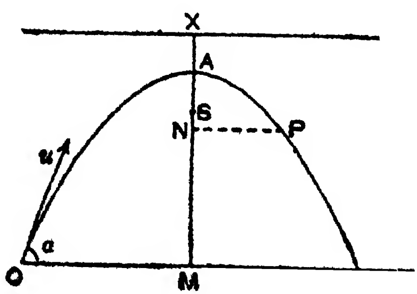
and

$$y = PM = \textit{vertical displacement in time } t$$
$$= u \sin \alpha . t - \tfrac{1}{2}gt^2. \qquad \cdots \qquad \text{(iv)}$$

(i) and (iii) give v and θ, *i.e.*, the velocity at P in magnitude and direction, whereas (ii) and (iv) give the position P at the instant.

8·5. The path of projectile (in vacuo) is a parabola.

First proof.



Let a particle be projected from O with a velocity u at an angle α with the horizon. Let A be the highest point of

the path described by the particle, and XAM the vertical through A. Let us measure time from the instant when the particle passes through A, t being considered positive towards the right and negative towards the left of A. At any instant t from A (positive or negative) let P be the position of the projectile, and PN the perpendicular from P on AM.

At A the vertical velocity of the particle is evidently zero, whereas, since there is no acceleration in the horizontal direction, the horizontal component of the velocity is constant throughout the motion and equal to its initial value $u \cos \alpha$. The acceleration vertically downwards is g due to gravity.

Thus,

PN = horizontal displacement of the particle in time $t = u \cos \alpha \,.\, t$

AN = vertical displacement from A in time t $= \frac{1}{2}gt^2$.

$$\therefore \quad \frac{PN^2}{AN} = \frac{u^2 \cos^2\alpha . t^2}{\frac{1}{2}gt^2} = \frac{2u^2 \cos^2\alpha}{g}$$

a constant independent of t, and therefore same for all positions of P on the path.

This identifies the locus of P to be a parabola, with vertex A, axis AM vertically downwards, and *latus rectum* $4AS = \frac{2u^2 \cos^2\alpha}{g} = \frac{2}{g}$ *(horizontal velocity)*^{2}.

Cor. If we measure $AS = \frac{u^2 \cos^2\alpha}{2g}$ along AM, S is evidently the focus of the parabola. Also, measuring $AX = AS$ above A, the horizontal through X is the directrix of the parabola.

The *height of the directrix above* $O = AM + AX$

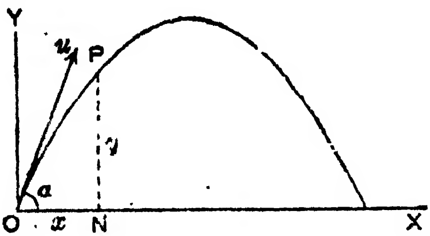
$$= \frac{u^2 \sin^2\alpha}{2g} + \frac{u^2 \cos^2\alpha}{2g} = \frac{u^2}{2g}, \text{ which is independent of } \alpha.$$

Height of the focus $= MS = MA - AS$

$$= \frac{u^2 \sin^2\alpha}{2g} - \frac{u^2 \cos^2\alpha}{2g} = -\frac{u^2}{2g} \cos 2\alpha.$$

Thus, *all trajectories described with the same velocity of projection u from the same point in different directions in a vertical plane are parabolas with a common directrix whose height above the point of projection is $\frac{u^2}{2g}$, the same as attained by a particle projected vertically upwards with the same initial velocity.*

Second proof.



Let a particle be projected from O with a velocity u at an angle a to the horizon. Let the horizontal and vertical through O in the plane of projection be chosen as axes of x and y respectively. At any instant t, let P be the position of the particle whose co-ordinates are $(x, y.)$

As there is no horizontal acceleration, the horizontal component of velocity of the particle is constant throughout the motion and equal to its initial value $u \cos a$

$$\therefore \quad x = u \cos a \, . \, t. \qquad \cdots \qquad \text{(i)}$$

Again, the initial vertical component of velocity is $u \sin a$, and the acceleration in the vertical direction is $-g$ due to gravity.

$$\therefore \quad y = u \sin a \, . \, t - \tfrac{1}{2}gt^2. \qquad \cdots \qquad \text{(ii)}$$

From (i) and (ii), eliminating t,

$$y = u \sin a \frac{x}{u \cos a} - \tfrac{1}{2}g\left(\frac{x}{u \cos a}\right)^2$$

$$= x \tan a - \frac{g}{2u^2 \cos^2 a} x^2 \qquad \cdots \qquad \text{(iii)}$$

which being of the form $y = Ax + Bx^2$, is the equation to a parabola. Hence, the locus of P is a parabola.

Note. The above equation of the trajectory can be written as

$$x^2-\frac{2u^2\sin\alpha\cos\alpha}{g}x=-\frac{2u^2\cos^2\alpha}{g}y,$$

i.e., $$\left(x-\frac{u^2\sin\alpha\cos\alpha}{g}\right)^2=-\frac{2u^2\cos^2\alpha}{g}\left(y-\frac{u^2\sin^2\alpha}{2g}\right).$$

Transferring the origin to the point $\left(\frac{u^2\sin\alpha\cos\alpha}{g}, \frac{u^2\sin^2\alpha}{2g}\right)$ the equation becomes $x^2=-\frac{2u^2\cos^2\alpha}{g}y$.

Hence, the path is a parabola whose vertex is the point $\left(\frac{u^2\sin\alpha\cos\alpha}{g}, \frac{u^2\sin^2\alpha}{2g}\right)$, whose latus rectum is $\frac{2u^2\cos^2\alpha}{g}$ and whose axis is vertical and drawn downwards. We easily get from above

co-ordinates of the vertex : $\left(\frac{u^2\sin 2\alpha}{2g}, \frac{u^2\sin^2\alpha}{2g}\right)$,

co-ordinates of the focus : $\left(\frac{u^2\sin 2\alpha}{2g}, -\frac{u^2\cos 2\alpha}{2g}\right)$.

Alternatively, these co-ordinates may be obtained from the values of OM, AM, SM (See *figure*, *Art.* 8·5).

Third proof.

Let a particle be projected from any point B with a velocity u in any direction BP, and let BV be the vertical through B.

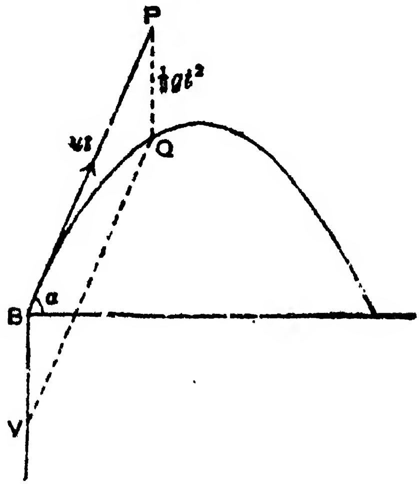
Had there been no gravity, the particle would move uniformly with the velocity u, and describing a distance ut along the direction of projection would reach the point P, say. Owing to the acceleration g due to gravity, however, assuming the initial velocity to be absent, the particle would receive a downward vertical displacement $\frac{1}{2}gt^2$. Hence, if from P we take a length $PQ=\frac{1}{2}gt^2$ vertically downwards, Q represents the actual position of the projectile at time t, taking into consideration the initial velocity and the acceleration due to gravity.

Now, QV being drawn parallel to BP, meeting BV at V,

$$QV = BP = ut$$

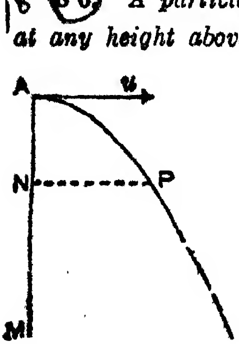
$$BV = PQ = \tfrac{1}{2}gt^2.$$

$$\therefore \quad \frac{QV^2}{BV} = \frac{u^2t^2}{\frac{1}{2}gt^2} = \frac{2u^2}{g}, \text{ a constant}$$



which, since B is a fixed point, and QV and BV are in fixed directions, identifies the locus of Q to be a parabola.

86. *A particle is projected* **horizontally** *from a point at any height above the ground; to show that the path described by it is a parabola.*



Let a particle be projected horizontally with a velocity u from a point A, and let AM be the downward vertical through A. Let P be the position of the particle at any time t, and let PN be drawn perpendicular to AM.

Then since there is no horizontal acceleration, the horizontal velocity

remains unchanged throughout the motion, namely, u. Also, at A the vertical velocity is zero, and due to gravity the vertical acceleration during the motion is g downwards.

Thus, PN = horizontal displacement in time t

$= ut$

and AN = vertical displacement in time t

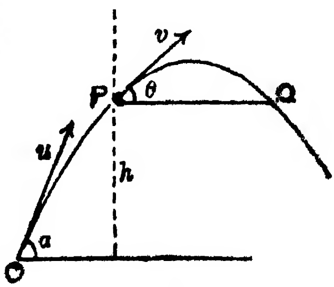
$= \frac{1}{2}gt^2$.

$$\therefore \quad \frac{PN^2}{AN} = \frac{u^2t^2}{\frac{1}{2}gt^2} = \frac{2u^2}{g}, \text{ a constant,}$$

showing that the locus of P is a parabola with A as vertex, AM as axis, and $\frac{2u^2}{g}$ as latus rectum.

8·7. *To find the velocity of a projectile at any point of its path at a given height from the point of projection and to show that the magnitude of the velocity is the same as that acquired by a particle allowed to fall vertically from the directrix to the point.*

Let a particle be projected from O with a velocity u at an angle a to the horizon and let v be its velocity at an angle θ to the horizon, at a point P of its path whose height is h above O.



As there is no horizontal acceleration, the horizontal component of velocity remains unchanged throughout the motion, and so

$$v \cos \theta = u \cos a. \quad \cdots \text{(i)}$$

Again, the initial upward vertical component of velocity is $u \sin a$ and $-g$ is the acceleration due to gravity in that direction. Hence, considering the motion in the vertical direction,

$$v^2 \sin^2\theta = u^2 \sin^2 a - 2gh. \quad \cdots \text{(ii)}$$

From (i) and (ii), $\mathbf{v}^2 = \mathbf{u}^2 - 2\mathbf{gh}$ ... (iii)

and $\tan\theta = \pm \dfrac{\sqrt{u^2 \sin^2 a - 2gh}}{u \cos a}$ ... (iv)

giving the *velocity at height h* in magnitude and direction.

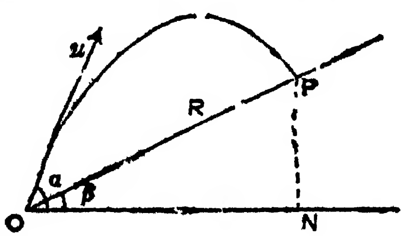
Again, since the height of the directrix of the trajectory above O is known to be $\frac{u^2}{2g}$, a particle allowed to fall freely from the directrix to the point P describes a vertical distance $\frac{u^2}{2g} - h$, and since it moves with acceleration g due to gravity, it acquires in that time a velocity v' given by

$$v'^2 = 2g\left(\frac{u^2}{2g} - h\right) = u^2 - 2gh = v^2.$$

$\therefore$ $\mathbf{v}' = \mathbf{v}$. ... ... (v)

Note Considering the magnitude of v with positive sign, the double sign of $\tan\theta$ correspond to the two positions P and Q of the particle at the same height h above O, once while rising and once while falling. Thus at these points the directions of motion make equal angles with the horizon, one above, and the other below it.

8·8. Range on an inclined plane and time of flight.



Let OP be an inclined plane of inclination β to the horizon through O, from which a particle is projected with a velocity u at an angle of projection a to the horizon, in the vertical plane through a line of greatest slope. Let the projectile, describing its path, meet the plane at P, so that OP ($= R$ say) is the range on the inclined plane. Let T be the corresponding time of flight.

Resolving the motion along and perpendicular to the plane, the initial component of velocity perpendicular to the plane (in the upward direction) is evidently $u \sin(\alpha-\beta)$. The acceleration due to gravity is $-g$ vertically upwards, and its component perpendicular to the plane (in the upward direction) is easily seen to be $-g \cos \beta$. After time T, the particle being again on the plane, displacement of the particle perpendicular to the plane is zero, and hence,

$$0 = u \sin(\alpha-\beta).T - \tfrac{1}{2}g \cos \beta.T^2.$$

(by the formula $s = ut + \frac{1}{2}ft^2$)

$$\therefore \quad T = \frac{2u \sin(\alpha-\beta)}{g \cos \beta}. \quad \ldots \quad \ldots \quad \text{(i)}$$

As there is no horizontal acceleration, the horizontal component of velocity during this motion is constant throughout, being equal to its initial values $u \cos \alpha$, and so the horizontal displacement

$$ON = u \cos \alpha . T, \qquad i.e., \quad R \cos \beta = u \cos \alpha . T.$$

$$\therefore \quad R = \frac{u \cos \alpha}{\cos \beta}.\ T = \frac{2u^2 \sin(\alpha-\beta) \cos \alpha}{g \cos^2\beta},$$

$$i.e., \quad R = \frac{u^2}{g \cos^2\beta}[\sin(2\alpha-\beta) - \sin \beta]. \quad \ldots \quad \text{(ii)}$$

Alternative method.

From the triangle OPN,

$$\tan \beta = \frac{PN}{ON} = \frac{\text{vertical displacement in time } T}{\text{horizontal displacement in time } T}$$

$$= \frac{u \sin \alpha.T - \frac{1}{2}gT^2}{u \cos \alpha T} = \tan \alpha - \frac{gT}{2u \cos \alpha}.$$

$$\therefore \quad T = \frac{2u \cos \alpha}{g}(\tan \alpha - \tan \beta) = \frac{2u \sin(\alpha-\beta)}{g \cos \beta} \quad \ldots \quad \text{(A)}$$

Again, $OP = ON \sec \beta = \dfrac{u \cos \alpha . T}{\cos \beta}$

$$= \frac{u \cos \alpha}{\cos \beta}.\frac{2u \cos \alpha}{g}(\tan \alpha - \tan \beta) \quad [\textit{from } A]$$

$$\therefore \quad \text{range } R = \frac{2u^2 \cos^2 \alpha}{g \cos \beta}(\tan \alpha - \tan \beta)$$

$$= \frac{2u^2 \cos \alpha \sin (\alpha - \beta)}{g \cos^2 \beta}.$$

Note. For *another method*, see Art. 9·3, result (28).

Cor. 1. Maximum Range on an inclined plane.

From the above value of R, with given u, the range on a given inclined plane is maximum when $\sin (2\alpha - \beta)$ is greatest, that is, unity.

Hence,

$$R_{max} = \frac{u^2}{g \cos^2 \beta}(1 - \sin \beta) = \frac{u^2}{g(1 + \sin \beta)} \quad \cdots \quad \text{(iii)}$$

and this occurs when $2\alpha - \beta = \frac{\pi}{2}$, or, $\alpha = \frac{1}{2}\left(\frac{\pi}{2} + \beta\right)$.

Thus, for *maximum range* on an inclined plane *the direction of projection bisects* the angle between the horizontal and the normal to the plane, or what amounts to the same thing, *the angle between the plane and the vertical.*

Cor. 2. *For a given range on an inclined plane with a given velocity of projection, there are two directions of projection, equally inclined to the direction of maximum range.*

From result (ii) above,

$$\sin (2\alpha - \beta) = \sin \beta + \frac{g \cos^2 \beta}{u^2} R.$$

Hence, with R, u, β given, the right-hand side is known, and thus an acute angle θ of which the quantity (which is clearly positive) is the sine, can be determined from the Trigonometrical tables. Hence,

$$2\alpha - \beta = \theta, \qquad \text{or,} \qquad \pi - \theta.$$

$$\therefore \quad \alpha = \frac{\theta}{2} + \frac{\beta}{2}, \qquad \text{or,} \qquad \frac{\pi}{2} - \frac{\theta}{2} + \frac{\beta}{2},$$

giving two possible values of α (within the range 0 to π) *i.e.*, two possible directions of projection.

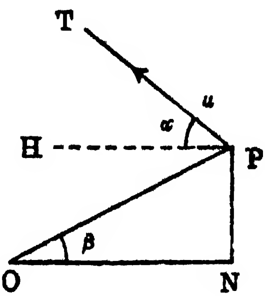
Again, the angle of projection for the maximum range on the plane is $\frac{\pi}{4}+\frac{\beta}{2}$, and since

$$\left(\frac{\pi}{4}+\frac{\beta}{2}\right)-\left(\frac{\theta}{2}+\frac{\beta}{2}\right)=\left(\frac{\pi}{2}-\frac{\theta}{2}+\frac{\beta}{2}\right)-\left(\frac{\pi}{4}+\frac{\beta}{2}\right)=\frac{\pi}{4}-\frac{\theta}{2},$$

the latter part of the result follows.

8·8 (A). Range down an inclined plane and the time of flight.

Suppose a particle is projected from the point P with velocity u, at an angle α to the horizontal PH.



Then its velocity along PO (*i.e.*, down the plane) is $u\cos(\alpha+\beta)$ and an acceleration $g\sin\beta$ along PO and its velocity perpendicular to PO in the upward direction is $u\sin(\alpha+\beta)$ and an acceleration $g\cos\beta$ perpendicular to PO in the downward direction. Let T' be the time (*down* the plane) *i.e.*, from P to O. Since after time T', the particle is again on the plane, the displacement of the particle, perpendicular to the plane is zero.

$$\therefore \quad 0=u\sin(\alpha+\beta)T'-\tfrac{1}{2}g\cos\beta\, T'^2$$

$$\therefore \quad T'=\frac{2u\sin(\alpha+\beta)}{g\cos\beta}. \qquad \cdots \qquad \cdots \quad (1)$$

Similarly, it can be shown as in Art. 8·8 that if R' be the range *down* the plane (PO), then

$$R'=\frac{2u^2\cos\alpha\sin(\alpha+\beta)}{g\cos^2\beta}$$

$$=\frac{u^2}{g\cos^2\beta}\Big[\sin(2\alpha+\beta)+\sin\beta\Big]. \qquad \cdots \quad (2)$$

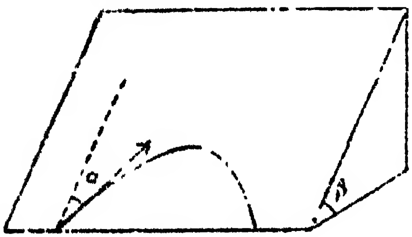
Similarly, we obtain as in Cor. 1 of Art. 8·8, the maximum value of R'

$$R'_{max} = \frac{u^2}{g(1-\sin\beta)}. \quad \ldots \quad \ldots \quad (3)$$

It is evident that *the values given in (1), (2), (3) are deducible from the corresponding values of (i), (ii), (iii) of Art. 8·8 and Cor. 1 by changing β to $-\beta$.*

8·9. Motion upon a smooth inclined plane.

Let a particle on a smooth inclined plane be given a sliding velocity u in a direction making an angle α with the line of greatest slope. Now it has been proved in a previous chapter (See *Art. 5·6*) that for a particle moving on a smooth inclined plane, the only effective acceleration on account of gravity is $g \sin\beta$ down the plane along the line of greatest slope, where β is the inclination of the plane to the horizon. Hence, resolving along the line of greatest slope (upwards) and perpendicular to it, the initial components of velocity are respectively $u\cos\alpha$ and $u\sin\alpha$, and the corresponding components of acceleration are $-g\sin\beta$ and zero. Considering the motion along these two directions separately, with the help of the usual formulæ for uniformly accelerated motion, any problem on sliding motion of the particle on the smooth inclined plane can be worked out.



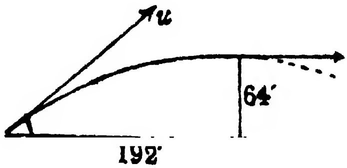
8·10. Illustrative Example.

Ex. 1. *A shot after leaving a gun passes just over a wall of a fort horizontally. If the wall is 64 ft. high and 192 ft. distant from the gun, find the direction and velocity of projection of the shot.* [*U. P. 1937*]

Let u and α be the velocity and angle of projection of the shot, and t secs. be the time after which the shot passes over the wall.

As the vertical component of motion is zero then, we get

$$u \sin \alpha - gt = 0. \qquad \cdots \qquad \text{(i)}$$



Also considering horizontal and vertical displacement during the time,

$$u \cos \alpha \,.\, t = 192 \qquad \cdots \qquad \text{(ii)}$$

and $\quad u \sin \alpha \,.\, t - \frac{1}{2}gt^2 = 64,$

i.e., using (i), $\quad \frac{1}{2}gt^2 = 64,$

whence, $t^2 = 4$, or, $t = 2. \qquad \cdots \qquad$ (iii)

From (i) and (ii) now,

$$u \sin \alpha = gt = 64$$

$$u \cos \alpha = 192/t = 96.$$

$$\therefore \quad u = \sqrt{64^2 + 96^2}$$

$$= 32\sqrt{13} \text{ ft./sec.}$$

Also, $\quad \tan \alpha = 64/96 = \frac{2}{3},$

or, $\quad \alpha = \tan^{-1} \frac{2}{3}.$

Otherwise :

It might be noted that the motion being horizontal at the top of the tower, this point is the highest point of the trajectory and thus we can use the known results,

$$\frac{u^2 \sin^2 \alpha}{2g} = 64$$

and $\quad 192 = \text{half the range} = \dfrac{u^2 \sin \alpha \cos \alpha}{g}.$

These also would give the values of α and u as before.

Ex. 2. *Two shots are fired simultaneously from the same point with velocities in the ratio of 13 : 5 $\sqrt{5}$, and they hit the same mark on the horizontal plane through the point of projection at a distance 540 feet from it. If the greatest heights attained by the shots be in the ratio of 5 : 9, find the time interval between their hitting the mark.*

Let u and u' be velocities of projection of the shots and α, α' their respective angles of projection. Then from the given conditions

$$\frac{u}{u'}=\frac{13}{5\sqrt{5}}. \quad \ldots \quad \ldots \quad \text{(i)}$$

$$\frac{2u^2 \sin\alpha\cos\alpha}{g}=540=\frac{2u'^2\sin\alpha'\cos\alpha'}{g} \quad \ldots \quad \text{(ii)}$$

and $\quad \dfrac{u^2\sin^2\alpha}{2g} : \dfrac{u'^2\sin^2\alpha'}{2g}=5:9$, whence,

$$\frac{u\sin\alpha}{\sqrt{5}}=\frac{u'\sin\alpha'}{3}=K \text{ (say)}. \quad \ldots \quad \text{(iii)}$$

Thus, from (ii), $u\cos\alpha=\dfrac{540\times 16}{K\sqrt{5}}, \quad u\cos\alpha'=\dfrac{540\times 16}{3K}$

and therefore, using (iii),

$$u^2=5K^2+\frac{(540\times 16)^2}{5K^2}, \qquad u'^2=9K^2+\frac{(540\times 16)^2}{9K^2}.$$

Thus, from (i),

$$\frac{169}{125}=\frac{u^2}{u'^2}=\frac{9}{5}\cdot\frac{25K^4+(540\times 16)^2}{81K^4+(540\times 16)^2},$$

whence $\quad 169\{81K^4+(540\times 16)^2\}=225\{25K^4+(540\times 16)^2\}$,

$$\text{giving } K^4=\frac{(540\times 16)^2(225-169)}{81\times 169-25\times 225}=\frac{(540\times 16)^2}{9\times 16}=(180\times 4)^2.$$

$\therefore$ $K^2=180\times 4$, or $K=12\sqrt{5}$, the positive value of K being taken since by (iii), $u\sin\alpha=K\sqrt{5}$ represents the initial upward vertical velocity of the first shot which is clearly positive.

Now the times of flight of the shots are $\dfrac{2u\sin\alpha}{g}$ and $\dfrac{2u'\sin\alpha'}{g}$, and since the shots start simultaneously, the required time interval between their hitting the mark

$$=\frac{2u'\sin\alpha'}{g}-\frac{2u\sin\alpha}{g}=\frac{2}{g}(3K-K\sqrt{5})$$

$$=\frac{12\sqrt{5}}{16}(3-\sqrt{5})=\frac{3}{4}(3\sqrt{5}-5)\text{ seconds.}$$

Ex 3. *A particle is projected from a point O so as to pass through two given points in the same vertical plane with O, at heights h_1 and h_2 above O, and at horizontal distances d_1 and d_2 from it on the same side. Find the angle of projection.*

Let u and α be the velocity and the angle of projection respectively.

Let t_1 be the time to reach the first point.

Considering the horizontal and vertical motions separately in the case,

$$u \cos \alpha \,.\, t_1 = d_1$$

$$u \sin \alpha \,.\, t_1 - \tfrac{1}{2} g t_1^2 = h_1.$$

From these, eliminating t_1,

$$d_1 \tan \alpha - \tfrac{1}{2} g \frac{d_1^2}{u^2 \cos^2 \alpha} = h_1,$$

$$\text{or,} \quad \frac{\tan \alpha}{d_1} - \frac{1}{2} \frac{g}{u^2 \cos^2 \alpha} = \frac{h_1}{d_1^2}.$$

Similarly, from the other case,

$$\frac{\tan \alpha}{d_2} - \frac{1}{2} \frac{g}{u^2 \cos^2 \alpha} = \frac{h_2}{d_2^2}.$$

$$\therefore \quad \text{subtracting, } \tan \alpha \left(\frac{1}{d_2} - \frac{1}{d_1} \right) = \frac{h_2}{d_2^2} - \frac{h_1}{d_1^2},$$

$$\therefore \quad \alpha = \tan^{-1} \left(\frac{d_1^2 h_2 - d_2^2 h_1}{d_1 d_2 (d_1 - d_2)} \right).$$

Ex. 4. *A particle is projected from the base of a hill whose shape is that of a right circular cone with axis vertical. The projectile grazes the vertex and strikes the hill again at a point on the base. If α be the semi-vertical angle of the cone, h its height, V the initial velocity of the projectile and θ the angle of projection measured from the horizontal, show that*

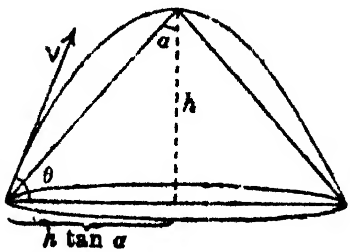
$$\tan \theta = 2 \cot \alpha$$

and $$V^2 = gh\left(2 + \tfrac{1}{2} \tan^2 \alpha\right).$$

Let t be the time to reach the vertex. Then considering horizontal and vertical displacements in this time, we get

$$V\cos\theta\,.\,t = h\tan\alpha$$

$$V\sin\theta\,.\,t - \tfrac{1}{2}gt^2 = h.$$



Eliminating t,

$$h\tan\alpha\tan\theta - \tfrac{1}{2}g\,.\,\frac{h^2\tan^2\alpha}{V^2\cos^2\theta} = h,$$

or, $$\tan\theta - \frac{gh\tan\alpha}{2V^2\cos^2\theta} = \cot\alpha. \quad \cdots \quad \text{(i)}$$

Again, the horizontal range of the projectile is easily seen from the figure,

$$2h\tan\alpha = \frac{2V^2\sin\theta\cos\theta}{g},$$

or, $$\frac{gh\tan\alpha}{V^2\cos\theta} = \sin\theta. \quad \cdots \quad \text{(ii)}$$

Thus, from (i), (ii) eliminating V^2,

$$\tan\theta - \tfrac{1}{2}\tan\theta = \cot\alpha,$$

or, $$\tan\theta = 2\cot\alpha.$$

Again from (ii),

$$V^2 = \frac{gh\tan\alpha}{\sin\theta\cos\theta} = \frac{gh\tan\alpha\,.\,\sec^2\theta}{\tan\theta}$$

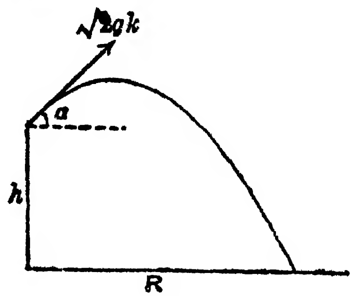
$$= \frac{gh\tan\alpha\,(1+4\cot^2\alpha)}{2\cot\alpha}$$

$$= gh(2 + \tfrac{1}{2}\tan^2\alpha).$$

Ex. 5. *A fort and a ship are both armed with guns which can fire with a muzzle velocity of $\sqrt{2gk}$, and the guns in the fort are at a height h above the guns in the ship. If d_1 and d_2 are the greatest horizontal ranges at which the fort and the ship can respectively engage, prove that*

$$\frac{d_1}{d_2}=\sqrt{\frac{k+h}{k-h}}.$$

Let a be the angle of projection of a gun in the fort, and let the shot reach the horizontal plane through the ship at time t, at a horizontal range R.



Then

$$\sqrt{2gk}\cdot\cos a\,.\,t=R \qquad \cdots \quad \text{(i)}$$

$$\sqrt{2gk}\cdot\sin a\,.\,t-\tfrac{1}{2}gt^2=-h \quad \text{(ii)}$$

First Method.

Eliminating a between (i) and (ii),

$$2gk\,.\,t^2=R^2+(\tfrac{1}{2}gt^2-h)^2,$$

$$\text{or,}\quad R^2=g(2k+h)\,t^2-h^2-\tfrac{1}{4}g^2t^4$$

$$=\{(2k+h)^2-h^2\}-\{\tfrac{1}{2}gt^2-(2k+h)\}^2$$

and clearly, the maximum value of the right-hand side occurs when the last perfect square term involving the variable t is zero. As d_1 is the given maximum value of R, we get

$$d_1{}^2=(2k+h)^2-h^2=4k(k+h).$$

Exactly in a similar way, considering projectiles fired from the ship reaching the level of the fort, (by replacing h by $-h$ in this case),

$$d_2{}^2=4k(k-h).$$

Hence, $\dfrac{d_1{}^2}{d_2{}^2}=\dfrac{k+h}{k-h}$, or, $\dfrac{d_1}{d_2}=\sqrt{\dfrac{k+h}{k-h}}.$

Second Method. **(*First case*).**

Eliminating t between (i) and (ii), we get

$$-h=R\tan a-\tfrac{1}{4}g\frac{R^2}{gk}(1+\tan^2 a)$$

$$i.e.,\ \tan^2 a-\frac{4k}{R}\tan a+\left(1-\frac{4kh}{R^2}\right)=0.$$

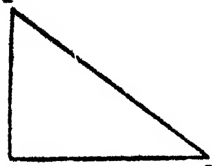
Since the quadratic in $\tan\alpha$ has real roots,

$$\therefore \qquad \left(1-\frac{4kh}{R^2}\right) \not< \frac{4k^2}{R^2},$$

$$\text{i.e., } R^2 \not< 4k(h+k), \quad \text{i.e., } R \not< \sqrt{4k(h+k)}.$$

$\therefore$ the greatest horizontal range $d_1 = 2\sqrt{\{k(h+k)\}}$.

Third Method. (*Second case*).


O S F

Let F be the fort and S be the ship from where the fort can be attacked. Let $\angle OSF = \beta$, and $OF = h$. The maximum range up the plane is SF. Let $SF = R$.

$$\therefore \qquad R = \frac{2gk}{g\left(1+\frac{h}{R}\right)}$$

(*by the formula of Art.* 8 8, *Cor.* 1)

$$\therefore \quad R = 2k - h.$$

$\therefore$ the maximum horizontal range SO i.e., $d_2 = \sqrt{(R^2 - h^2)}$

$$= \sqrt{\{(2k-h)^2 - h^2\}} = 2\sqrt{\{k(k-h)\}}.$$

(*1st case*). If R' be the maximum range down the plane, then we get from Art. 8·8 (A),

$$R' = \frac{2gk}{g\left(-\frac{h}{R'}\right)}$$

$$\text{i.e.,} \qquad R' = 2k + h.$$

Now proceeding as in the 1st case, required maximum horizontal range is obtained.

Ex. 6. *A particle projected with a velocity u, strikes at right angles a plane through the point of projection inclined at an angle β to the horizon. Show that the time of flight is $\dfrac{2u}{g\sqrt{1+3\sin^2\beta}}$. Find also the height of the point struck above the point of projection.* [C.H. 1962, '70]

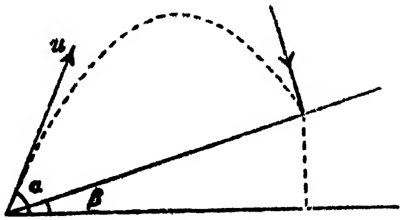
Let t be the required time of flight, and α the angle of projection with the horizontal.

Resolving the motion along the perpendicular to the inclined plane, we get from the given conditions in this case,

$$u \cos(\alpha-\beta) - g \sin\beta \, . \, t = 0, \qquad \cdots \quad \text{(i)}$$

$$u \sin(\alpha-\beta) \, . \, t - \tfrac{1}{2} g \cos\beta \, . \, t^2 = 0,$$

i.e.,
$$u \sin(\alpha-\beta) - \tfrac{1}{2} g \cos\beta \, . \, t = 0. \qquad \cdots \quad \text{(ii)}$$



From (i) and (ii), eliminating $(\alpha-\beta)$,

$$u^2 = g^2 t^2 (\sin^2\beta + \tfrac{1}{4}\cos^2\beta) = \tfrac{1}{4} g^2 t^2 (1 + 3 \sin^2\beta).$$

$$\therefore \quad t = \frac{2u}{g\sqrt{1+3\sin^2\beta}}. \qquad \cdots \quad \text{(iii)}$$

Again, as the horizontal distance of the point struck is $u \cos\alpha \, . \, t$, the height of this point above the point of projection is

$$u \cos\alpha \, . \, t \, . \tan\beta = ut \tan\beta \, . \cos(\alpha-\beta+\beta)$$

$$= t \tan\beta \, \{u \cos(\alpha-\beta) \, . \cos\beta - u \sin(\alpha-\beta) \, . \sin\beta\}$$

$$= t \tan\beta \, \{g \sin\beta \, . \, t \, . \cos\beta - \tfrac{1}{2} g \cos\beta \, . \, t \, . \sin\beta\}$$

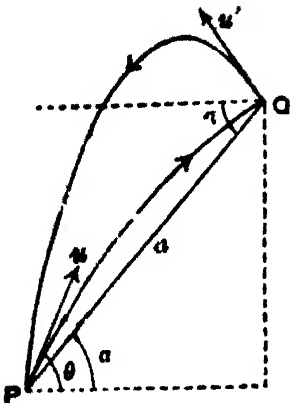
[*using (i) and (ii)*]

$$= \tfrac{1}{2} g t^2 \sin^2\beta = \frac{2u^2}{g} \cdot \frac{\sin^2\beta}{1+3\sin^2\beta} \quad [\textit{using (iii)}]$$

Ex. 7. *The line joining P to Q is inclined at an angle α to the horizontal. Show that the least velocity required to shoot from P to Q is to the least velocity required to shoot from Q to P as*

$$(\cos \tfrac{1}{2}\alpha + \sin \tfrac{1}{2}\alpha) : (\cos \tfrac{1}{2}\alpha - \sin \tfrac{1}{2}\alpha).$$

Assuming a to be the distance PQ which is inclined at an angle α to the horizon, the horizontal distance between P and Q, is $a \cos \alpha$, and the vertical height of Q above P is $a \sin \alpha$.



Let u be the velocity of a shot fired from P at an angle θ to the horizon, which reaches Q after a time t seconds.

Then, $\quad u \cos \theta \,.\, t = a \cos \alpha,$

and $\quad u \sin \theta \,.\, t - \frac{1}{2} g t^2 = a \sin \alpha.$

Hence, eliminating θ,

$$u^2 t^2 = a^2 \cos^2 \alpha + (a \sin \alpha + \tfrac{1}{2} g t^2)^2,$$

or,
$$u^2 = \tfrac{1}{4} g^2 t^2 + g a \sin \alpha + \frac{a^2}{t^2}$$
$$= \left(\frac{1}{2} g t - \frac{a}{t}\right)^2 + g a\,(1 + \sin \alpha).$$

Now t being variable for different values of the velocity of projection u, the square term of the right-hand side has its least value zero. Hence, the least possible value of the velocity of projection from P, so that the shot may reach Q, is given by

$$u^2 = g a\,(1 + \sin \alpha) = g a \left(\cos \frac{\alpha}{2} + \sin \frac{\alpha}{2}\right)^2,$$

or, $$u = \sqrt{g a}\left(\cos \frac{1}{2} \alpha + \sin \frac{1}{2} \alpha\right).$$

Again, the line from Q to P is inclined at an angle $-\alpha$ to the horizon. Hence, exactly as above, replacing α by $-\alpha$, the least velocity of projection from Q, so that the shot may reach P, is given by

$$u' = \sqrt{g a}\left(\cos \frac{1}{2} \alpha - \sin \frac{1}{2} \alpha\right).$$

Thus, $\quad u : u' = (\cos \frac{1}{2}\alpha + \sin \frac{1}{2}\alpha) : (\cos \frac{1}{2}\alpha - \sin \frac{1}{2}\alpha).$

Ex. 8. *A shell bursts on contact with the ground and pieces from it fly in all directions with all velocities up to 80 feet per second. Show that a man 100 feet away is in danger for* $\frac{5}{2}\sqrt{2}$ *seconds.*

Suppose a shot, which flies with a velocity v at an angle θ to the horizon, hits the man after t seconds.

Then, $\quad v \cos \theta . t = 100$

$$v \sin \theta . t - \tfrac{1}{2} g t^2 = 0,$$

whence $\quad v^2 t^2 = 100^2 + \tfrac{1}{4} g^2 t^4, \quad$ or, $\quad \tfrac{1}{4} g^2 t^4 - v^2 t^2 + 100^2 = 0.$

$$\therefore \quad t^2 = \frac{v^2 \pm \sqrt{v^4 - 100^2 g^2}}{\frac{1}{2} g^2}.$$

With any velocity of projection v therefore, (such that $v^4 \nless 100^2 g^2$, *i.e.*, $v \nless 10\sqrt{g}$) there are two pieces that will hit the man, one hitting it earlier at time t_1, given by $t_1^2 = \frac{v^2 - \sqrt{v^4 - 100^2 g^2}}{\frac{1}{2} g^2}$, and the other at a later time t_2 given by $t_2^2 = \frac{v^2 + \sqrt{v^4 - 100^2 g^2}}{\frac{1}{2} g^2}$.

Considering the later times for different velocities of projection, we see that the latest time at which a shot can hit the man is when v is greatest, namely 80 ft./sec. and the corresponding time is given by

$$t_2^2 = \frac{80^2 + \sqrt{80^4 - 100^2 . 32^2}}{\frac{1}{2} . 32^2} = \frac{6400 + 3200\sqrt{3}}{32 \times 16},$$

$$\therefore \quad t_2 = \frac{10}{4}\sqrt{2 + \sqrt{3}} = \frac{5\sqrt{2}}{4}(\sqrt{3} + 1) \text{ seconds.}$$

After this time no shot can hit the man.

Again, considering the earlier times for the different velocities of projection, given by

$$t_1^2 = \frac{v^2 - \sqrt{v^4 - 100^2 g^2}}{\frac{1}{2} g^4} = \frac{2 . 100^2}{v^2 + \sqrt{v^4 - 100^2 g^2}},$$

it is clear from the last form that t_1 is least when v is greatest, namely 80. Hence, of all the pieces that hit the man, the earliest time when a piece hits him is given by

$$t_1^2 = \frac{80^2 - \sqrt{80^4 - 100^2 g^2}}{\frac{1}{2} g^2},$$

whence as before $t_1 = \frac{5\sqrt{2}}{4}(\sqrt{3} - 1)$ seconds and before this time no shot hits him, so that he has no danger earlier.

Hence, the man remains in danger for

$$\frac{5\sqrt{2}}{4}(\sqrt{3} + 1) - \frac{5\sqrt{2}}{4}(\sqrt{3} - 1) = \frac{5}{2}\sqrt{2} \text{ seconds.}$$

Examples on Chapter VIII

1. A boy can throw a ball 40 yds. vertically upwards; find the greatest horizontal distance he can throw it. How long will it be in air in the second case?

2. A cricket-ball struck from the ground pitches 100 yds. ahead after rising to a maximum height of $56\frac{1}{4}$ ft. Find the time of flight, and the direction in which it is struck

3. A projectile thrown from a point in a horizontal plane comes back to the plane in 4 seconds at a distance of 64 yds. from the point of projection. Find the velocity of projection in feet per sec. [*C. U. 1913*]

4. A bombshell on striking the ground (supposed to be horizontal) bursts, scattering its fragments with velocites of magnitude u in different directions; find the area of the ground covered by the fragments.

[*B. H. U. 1933; C. U. 1938*]

5. A stone is dropped from a balloon moving horizontally with a velocity 96 ft./sec. and reaches the ground in 4 seconds. Find the height of the balloon, and the velocity of the stone on striking the ground.

6. A ball is projected from the ground at an elevation of $\cos^{-1}\frac{3}{5}$ with a velocity 100 ft./sec. Find the distance of the ball from the point of projection after 2 seconds.

7. A tennis ball served horizontally from a height 6·25 ft. strikes the ground at a point 60 feet away from the server. If it just touches the net 40 feet away from the server, find the height of the net. [*U. P. 1941*]

8. A football is kicked, and just passes over a bar 12 ft. high and 20 feet away. Find the direction in which the ball is kicked, if the velocity generated by the kick is 40 ft./sec.

9. A boy running at a uniform speed of 10 ft./sec., throws up a ball vertically relative to himself and catches

it 10 ft. from the point where he threw it up. With what velocity relative to himself does he throw the ball ?

10. A cricket-ball thrown from a height of 6 feet, at an angle of 30° with the horizon, with a speed of 60 feet per second, is caught by another fieldsman, at a height of 2 feet from the ground. How far apart were the two men ?
[*U. P. 1940*]

11. A particle is shot from the ground and grazes the tops of two posts at heights 36 ft. and 64 ft. which stand 96 ft. horizontally apart. If the time from post to post be 5 seconds, find the initial velocity (magnitude) of the projectile.

12. A gun is fired at an elevation $\tan^{-1}\frac{1}{20}$ towards a person on the same horizontal plane as the gun. If the shot and the sound of the gun reach him at the same instant, find the range, the velocity of sound being 1120 ft./sec.

13. (i) The maximum range of a rifle bullet is 1200 yards. If the rifle is fired with the same elevation from a truck running at 15 miles per hour towards the target, prove that the range is increased by 110 yards.

(ii) A gun is fired from a moving horizontal platform, and the horizontal ranges of the shot are observed to be R and S when the platform is moving forwards and backwards respectively with velocity V. Prove that the elevation of the gun is

$$\tan^{-1}\left[\frac{g(R-S)^2}{4V^2(R+S)}\right]. \qquad [\textit{ C. H. 1967, '69 }]$$

14. A particle is projected so as to pass through two points whose horizontal distances from the point of projection are 36 and 72 ft., and which are at vertical heights 11 and 14 ft. above the horizontal plane through the point of projection. Find the velocity and direction of projection.

15. A ball is projected with a velocity of 64 ft. per sec. from the top of a tower 128 ft. high, at an elevation of 30°. Find when, where and with what velocity it will strike the ground.

16. A ball slides from rest down a smooth sloping roof of length 8 ft. and inclination of 60° to the vertical and then falls to the ground. If the top of the roof be at a height 12 ft. from the ground, find the distance of the point where the ball reaches the ground from the foot of the vertical wall passing through the top of the roof.

17. A fountain jet projects streams in all directions with a velocity of 12 ft. per sec. from a point 4 ft. above the centre of a circular basin. What must be the diameter of the basin to catch all the water ?

18. For a trajectory, show that the focus of the path lies above, on, or below the horizontal line through the point of projection, according as the angle of projection is greater than, equal to, or less than $\frac{1}{4}\pi$.

19. A shot fired at a mark in the horizontal plane through the point of projection goes a ft. beyond it when the angle of elevation is α. When the angle of elevation is β, it falls b ft. short of the mark. Show that the proper elevation to hit the mark is

$$\frac{1}{2}\sin^{-1}\left(\frac{a\sin 2\beta + b\sin 2\alpha}{a+b}\right).$$

20. A particle is projected with a velocity v at an angle of elevation α from a point on a horizontal plane. If R be the range, T the time of flight, and H the maximum height attained by the particle, prove that

$$g^2T^4 - 4T^2v^2 + 4R^2 = 0,$$

and $$16gH^2 - 8v^2H + gR^2 = 0.$$ [*C. U. 1945*]

21. A body is projected so that on its upward path it passes through a point x ft. horizontally and y ft. vertically from the point of projection. If R ft. is the range on the horizontal plane through the point of projection, show that the angle of elevation of the projection is

$$\tan^{-1}\left(\frac{y}{x}\cdot\frac{R}{R-x}\right).$$ [*C. U. 1944*]

(ii) Prove that the equation to the path of a projectile may be written in the form

$$y = x \tan \alpha \, (R - x)/R,$$

where R is the range on the horizontal plane through the point of projection and α the angle of elevation of the projection. [*C. H. 1963, '68 ; C. P. 1967*]

[*This follows from equation (iii) of Art. 8·5 (2nd proof).*]

22. Two heavy particles are projected at elevations α, β in the same vertical plane at the same instant with equal velocities from two fixed points, and meet. Show that

$$\alpha + \beta = \text{const.}$$

23. A vertical rod PQ subtends an angle θ at a point O in the same horizontal plane as the foot of the rod. Two balls are projected at the same instant from O, in directions making angles α and β with the horizontal, so that the former strikes the top of the rod at the same moment that the latter strikes the bottom. Prove that

$$\tan \alpha - \tan \beta = \tan \theta.$$

24. For a given horizontal range, if α_1, α_2 are any two possible angles of projection and t_1, t_2 the corresponding times of flight, then

$$\frac{t_1^2 - t_2^2}{t_1^2 + t_2^2} = \frac{\sin (\alpha_1 - \alpha_2)}{\sin (\alpha_1 + \alpha_2)}.$$

25. (i) A body is projected at an angle α to the horizontal so as just to clear two walls of equal height a, at a distance $2a$ from each other. Show that the range is equal to $2a \cot \frac{1}{2}\alpha$. [*C. H. 1966*]

(ii) A ball is projected from a point O so as just to clear two walls, one of height a above the level of O and at a distance b from O, and the second of height b above the level of O and at a distance a from O. Show that the range on the horizontal plane through O is

$$\frac{a^2 + ab + b^2}{a + b}.$$ [*C. H. 1968*]

(iii) A projectile just goes over a wall of height h ft. at a distance c ft. from the point of projection O which lies on a horizontal plane through the foot of the wall and hits a mark at a height h ft. and distance $2c$ ft. from O. Prove that the initial velocity V is given by

$$4V^2 = g(4c^2 + 9h^2)/h.$$ [C. H. 1963]

26. The angular elevation of an enemy's position on a hill s ft. above the gun position is β. Show that in order to shell it, the projectile's velocity must not be less than

$$\sqrt{gs(1 + \operatorname{cosec} \beta)}.$$ [C. U. 1946]

27. Show that there are in general two directions in which a particle may be projected from a given point to hit another given point (h, k) and the minimum velocity of projection is

$$\sqrt{[g\{k + \sqrt{(h^2 + k^2)}\}]}.$$

28. Particles are projected simultaneously with velocities of magnitude V from a given point in different directions in the same vertical plane. Prove that after t seconds they all lie on a circle. [C. U. 1941, '53 ; C. H. 1966]

29. A particle is projected with an initial velocity u. If the greatest height attained by the particle be H, prove that the range R on the horizontal plane through the point of projection is

$$R = 4\sqrt{\left\{H\left(\frac{u^2}{2g} - H\right)\right\}}.$$ [C. U. 1940]

30. If several particles are projected in the same vertical plane from the same point with the same velocity, show that the foci of all the parabolic paths lie on a circle.

31. (a) Particles are projected from a given point in the same vertical plane at the same elevation. Show that the locus of (i) the vertices

and (ii) the foci

of the parabolas which they describe is each a straight line.

(*b*) A number of particles are projected simultaneously from the same point with velocities of the same magnitude V, in the same vertical plane in different directions. Find the locus of the vertices of their paths. [*C. H. 1966*]

32. A rocket on striking the ground bursts, scattering its fragments with a speed of 128 ft. per sec. in all directions. Show that fragments may fall at a point 384 ft. away from the point of bursting at an interval of 4 seconds.

33. If v_1 and v_2 be the velocities of a projectile at the ends of a focal chord of its path, and v be the constant horizontal velocity, show that

$$\frac{1}{v_1^2}+\frac{1}{v_2^2}=\frac{1}{v^2}.$$

[*Tangents at the ends of a focal chord of a parabola intersect at right angles on the directrix.*]

34. An aeroplane flying with a constant velocity v, at a constant height h, passes directly over a gun. When the elevation of the aeroplane above the horizontal plane is θ as seen from the gun, the gun is fired point blank at it. Show that the shot hits the aeroplane if $2(V\cos\theta-v)v\tan^2\theta=gh$, where V is the initial velocity of the shot, its path being parabolic. [*C. U. 1932*]

35. A shot is fired at an elevation of α at a bomber flying in a horizontal straight line with acceleration f. At the instant of firing, the bomber is directly overhead at a height h and has a velocity v. Prove that no velocity of projection will make the shot hit the bomber unless

$$\tan\alpha > \frac{\sqrt{f^2h^2+2ghv^2}-fh}{v^2}.$$

36. If two particles are simultaneously projected from the same point in the same vertical plane, the line joining them moves parallel to itself.

37. A boy can throw a ball vertically upwards to a height H ft. Show that he cannot clear a wall h ft. high at a distance d ft. from him unless

$$2H > h+\sqrt{h^2+d^2}.$$

38. A, B are two points distance d apart, and at heights h_1, h_2 above a given horizontal plane. Prove that the minimum velocity with which a particle must be projected from the plane so as to pass through A and B is

$$\sqrt{g(h_1+h_2+d)}.$$

39. A bird is sitting on the top of a building 72 feet high. At what angle of elevation should a person standing 360 feet from the foot of the building fire a shot with a velocity of 120 ft./sec. so as to hit the bird as soon as possible ?

40. A ball is projected, and a second ball also from the same point and in the same direction, with a velocity equal to the vertical component of the velocity of the first ball. Prove that the path of the second passes through the focus of the path of the first.

41. If t be the time in which a projectile reaches a point P of its path, and t' be the time from P till it strikes the horizontal plane through the point of projection, show that the height of P above the plane is $\frac{1}{2}gtt'$.

42. If h and h' be the greatest heights in the two paths of a projectile with a given velocity for a given range R, prove that

$$R=4\sqrt{hh'}.$$

43. Show that the product of the two times of flight from P to Q with a given velocity of projection is $2PQ/g$.

44. A gun is mounted on a cliff of height h above the sea-level. If u be the muzzle-velocity of the shot, prove that the maximum range d at sea-level measured from the foot of the cliff is, (in ft. sec. units),

$$d=\frac{u}{4}\left(\frac{u^2}{64}+h\right)^{\frac{1}{2}}$$

and the angle of projection α is given by

$$\tan\alpha=\frac{u^2}{32d}.$$ [C. U. 1933]

45. A particle P is projected at an angle α to the horizon with velocity V, and is subsequently met by a second particle Q, which is let fall from the directrix of the path of P at the instant of projection of P. Show that the distance of the point of projection of P from the straight line described by Q is

$$\frac{V^2 \cot \alpha}{2g}.$$ [C. U. 1934]

46. Two shots are projected from a gun at the top of a hill with the same velocity u at the angles of projection α and β respectively. If the shots strike the horizontal ground through the foot of the hill at the same point, show that the height h of the hill above the plane is given by

$$h = \frac{2u^2(1 - \tan \alpha \tan \beta)}{g(\tan \alpha + \tan \beta)^2}.$$ [C. U. 1935]

47. Two particles are projected with velocities u, u' at elevations α, α' from the same point, at the same time, in the same vertical plane. Prove that the difference between the times to the other point common to their paths is

$$\frac{2uu' \sin(\alpha - \alpha')}{g(u \cos \alpha + u' \cos \alpha')}.$$

48. If t be the time of describing any portion PQ of the parabolic path of a projectile, show that

$$t \propto (\tan \alpha - \tan \beta),$$

where α and β are the angles which the tangents at P and Q make with the horizon.

49. A ball is projected at an angle α to the horizontal up a plane which passes through the point of projection and is of elevation β. Show that it strikes the plane

(i) horizontally, if $\tan \alpha = 2 \tan \beta$,

(ii) normally, if $\cot \beta = 2 \tan(\alpha - \beta)$,

i.e., if $\tan \alpha = 2 \tan \beta + \cot \beta$.

50. A ball is projected with a velocity of 28 ft. per sec. up an inclined plane which passes through the point of

projection and which is of elevation 30°. The ball strikes the plane at right angles. Find the range on the plane.

51. If R_1 and R_2 be the maximum ranges (with a given velocity of projection) up and down an inclined plane respectively, prove that $\frac{1}{R_1}+\frac{1}{R_2}$ is independent of the inclination of the plane, and if R be the maximum horizontal range, (with the same velocity of projection), then R is the harmonic mean between R_1 and R_2.

52. A fort is on the edge of a cliff of height h and a ship is on the sea. The velocity of the shells projected by both is $\sqrt{2gk}$. Show that there is an annular region in which the fort is out of danger of the ship but the ship is within the range of the fort. Find the area of the annular region.

53. The radii of the front and hind wheels of a carriage are a and b, and c is the distance between their axle-trees; a particle of dust driven from the highest point of the hind wheel is observed to alight on the highest point of the front wheel. Show that the velocity of the carriage is

$$\sqrt{\frac{(c+a-b)(c-a+b)}{4(b-a)}g}.$$

ANSWERS

1. 240 ft.; $\sqrt{15}$ sec. 2. $3\frac{3}{4}$ secs.; $\tan^{-1}\frac{3}{4}$ with the horizon.

3. 80. 4. $\frac{\pi u^4}{g^2}$.

5. 256 ft.; 160 ft./sec. at an angle $\tan^{-1}\frac{4}{3}$ with the horizon.

6. 153·6 ft. 7. 3 ft. $5\frac{3}{8}$ inches.

8. Either 45°, or $\tan^{-1} 4$ with the horizon.

9. 16 ft./sec. 10. $60\sqrt{3}$ ft. 11. 100 ft./sec.

12. 3920 ft. 14. 78 ft./sec. at an angle $\tan^{-1}\frac{5}{12}$ with the horizon.

15. 4 secs.; $128\sqrt{3}$ ft. from the foot of the tower; $64\sqrt{3}$ ft./sec. at an angle 60° with the horizon.

16. $8\sqrt{3}$ ft. 17. 15 ft. 27. $\tan^{-1}\frac{u^2}{gx}$. 31. (b) Ellipse.

39. 45° 50. 14 ft. 52. $8\pi hk$.

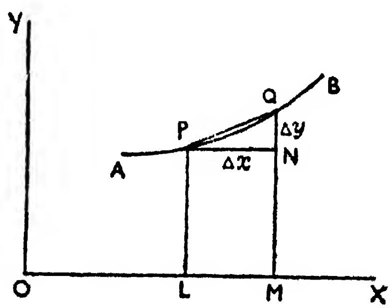
CHAPTER IX

TWO DIMENSIONAL MOTION

(*Analytical Treatment*)

9·1. When a particle is moving in a plane, the position of the particle can be determined if its co-ordinates (x, y) measured parallel to two fixed perpendicular lines OX, OY in the plane be known.

9·2. Components of Velocity and Acceleration along OX and OY.



Let the particle be moving along the curve APQ, and let it move from the position P at time t to the neighbouring position Q at time $t+\Delta t$.

Let the co-ordinates of P be (x, y) and those of Q be $(x+\Delta x,\ y+\Delta y)$ with reference to the rectangular axes OX, OY.

The displacement of the particle parallel to OX in time Δt is Δx.

the resolved part of the velocity parallel to OX, namely,

$$v_x = \underset{\Delta t \to 0}{Lt} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}.$$

Similarly, the resolved part of the velocity parallel to OY, namely,

$$v_y = \underset{\Delta t \to 0}{Lt} \frac{\Delta y}{\Delta t} = \frac{dy}{dt}.$$

Hence, if (x, y) be the co-ordinates of the point P at any instant with reference to a set of rectangular axes OX, OY, the component velocities parallel to the axes are the rates of change of x and y.

Thus, $$v_x = \frac{dx}{dt} \qquad \cdots \qquad \cdots \quad (1)$$

$$v_y = \frac{dy}{dt}. \qquad \cdots \qquad \cdots \quad (2)$$

The resultant velocity V is given by

$$V = \sqrt{(v_x^2 + v_y^2)} = \sqrt{\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right\}}. \qquad \cdots \quad (3)$$

Also, if θ be the angle which the direction of the resultant velocity (*i.e.*, the direction of motion) makes with the x-axis, we have

$$\tan\theta = \frac{dy/dt}{dx/dt} = \frac{dy}{dx}$$

i.e., the direction of motion at any point is the same as the direction of the tangent to the path at the point.

Similarly, components of acceleration f_x and f_y along OX, OY being the rates of changes of velocities along these directions, we have

$$f_x = \frac{d}{dt}(v_x) = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2} \qquad \cdots \quad (4)$$

$$f_y = \frac{d}{dt}(v_y) = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d^2y}{dt^2}. \qquad \cdots \quad (5)$$

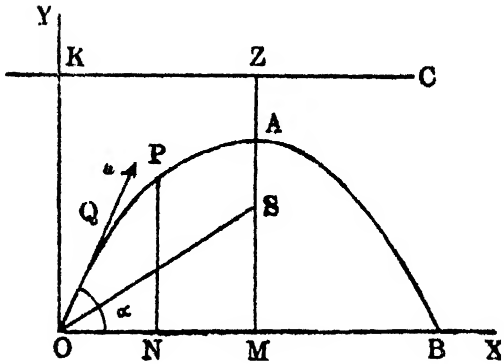
9·3. *A particle is projected with velocity u at an angle $\alpha(\neq \frac{1}{2}\pi)$ to the horizon ; to find its path.**

Let us take the starting point as origin and the axes of co-ordinates horizontal and vertical respectively.

Let (x, y) be the co-ordinates of the position P of the particle at any time t.

* Here, g is supposed constant, and resistance of air is neglected.

Since there is no force, and hence no acceleration in the horizontal direction, and since the vertical acceleration is



always the same, being equal to g downwards, we have here the equations of motion,

$$\frac{d^2x}{dt^2}=0, \quad \cdots (1) \qquad \frac{d^2y}{dt^2}=-g. \quad \cdots (2)$$

Integrating, $\dfrac{dx}{dt}=A \quad \cdots (3) \qquad \dfrac{dy}{dt}=-gt+B. \quad \cdots (4)$

Since, initially, *i.e.*, at time $t=0$, horizontal and vertical components of velocity are $u \cos a$, $u \sin a$, it follows from (2) and (4),

$$A=u \cos a, \; B=u \sin a. \quad \cdots (5)$$

Thus, (3) and (4) become

$$\frac{dx}{dt}=u \cos a \quad \cdots (6) \qquad \frac{dy}{dt}=u \sin a-gt. \quad \cdots (7)$$

Integrating again (6) and (7),

$$x=u \cos a.t+C \quad \cdots \quad \cdots (8)$$

$$y=u \sin a.t-\tfrac{1}{2}gt^2+D. \quad \cdots \quad \cdots (9)$$

Since, when $t=0$, $x=0$, $y=0$, we get from above $C=0$, $D=0$.

$\therefore$ (6) and (7) become

$$x = u\cos\alpha . t \qquad \cdots \qquad \cdots \qquad (10)$$

$$y = u\sin\alpha . t - \tfrac{1}{2}gt^2. \qquad \cdots \qquad \cdots \qquad (11)$$

From (10), $\quad t = \dfrac{x}{u\cos\alpha}$.

Substituting this value of t in (11), we get

$$y = x\tan\alpha - \tfrac{1}{2}g\,\frac{x^2}{u^2\cos^2\alpha} \qquad (12)$$

which is the equation of the path of the particle, and the equation being of the form $y = Ax + Bx^2$ (A and B being constants) it is the equation to a parabola.

Hence, *the path of the projectile is a parabola.*

Deductions :

(1) The equation (12) can be written as

$$x^2 - \frac{2u^2\sin\alpha\cos\alpha}{g}x = -\frac{2u^2\cos^2\alpha}{g}y$$

or, $$\left(x - \frac{u^2\sin\alpha\cos\alpha}{g}\right)^2 = -\frac{2u^2\cos^2\alpha}{g}\left(y - \frac{u^2\sin^2\alpha}{2g}\right).$$

or, $$\left(x - \frac{u^2\sin 2\alpha}{2g}\right)^2 = -\frac{2u^2\cos^2\alpha}{g}\left(y - \frac{u^2\sin^2\alpha}{2g}\right) \qquad \cdots \qquad (13)$$

Transferring the origin to the point $\left(\dfrac{u^2\sin 2\alpha}{2g}, \dfrac{u^2\sin^2\alpha}{2g}\right)$,

the equation becomes $X^2 = -\dfrac{2u^2\cos^2\alpha}{g}Y. \qquad \cdots \qquad (14)$

Hence, the *path is a parabola,* whose axis is *vertical* and *drawn downwards* and whose

vertex is $\left(\dfrac{u^2\sin 2\alpha}{2g}, \dfrac{u^2\sin^2\alpha}{2g}\right) \qquad \cdots \qquad (15)$

and *latus rectum* is $\dfrac{2u^2\cos^2\alpha}{g}. \qquad \cdots \qquad (16)$

(2) Putting $y=0$ in (12), we get either $x=0$ (*i.e.*, when the particle is at O), or $x=\frac{u^2 \sin 2\alpha}{g}$, $\left(i.e., \text{ when the particle is at } B, \text{ hence } OB=\frac{u^2 \sin 2\alpha}{g}\right)$. OB is called the *horizontal range*.

$$\therefore \quad \textbf{Horizontal Range} = \frac{u^2 \sin 2\alpha}{g}. \quad \cdots \quad (17)$$

This horizontal range is maximum when $\sin 2\alpha = 1$ *i.e.*, $\alpha = \frac{1}{4}\pi$ or 45°.

$$\therefore \quad \textbf{Maximum Horizontal Range} = \frac{u^2}{g}. \quad \cdots \quad (18)$$

Thus, for a given velocity of projection the horizontal range is maximum when the angle of projection is 45°.

(3). Putting $y=0$ in (11), we get either $t=0$ (*i.e.*, when the particle is at O at the time of start), or $t=\frac{2u \sin \alpha}{g}$ *i.e.*, when it is at B). The time from O to B is called time of flight. If T be the time of flight, then

$$T = \frac{2u \sin \alpha}{g}. \quad \cdots \quad (19)$$

(4) From (11), we see that y, *i.e.*, the height attained will be maximum when $\frac{dy}{dt}=0$ and $\frac{d^2y}{dt^2}$ is negative, *i.e.*, when $t=\frac{u \sin \alpha}{g}$ and substituting this value of t in (11) we have $y=\frac{u^2 \sin^2 \alpha}{2g}$. Thus, if T' denotes the time to the greatest height and H denotes the greatest height attained, then

$$T' = \frac{u \sin \alpha}{g} \quad \cdots \quad (20)$$

$$H = \frac{u^2 \sin^2 \alpha}{2g}. \quad \cdots \quad (21)$$

From (19) and (20), it is clear that the time from O to A is equal to the time from A to B.

Thus, ***time to the greatest height*** $=\frac{1}{2}$ ***(total time of flight).***

Since at the highest point, we have $\frac{dy}{dt}=0$, **so, the vertical component of the velocity is zero there. Hence,** ***at the highest point, the vertical component of velocity is zero.***

(5) From figure, height of the directrix

$$=MZ=MA+AZ$$

$$=\text{height of the vertex}+\tfrac{1}{4}\text{ (latus rectum)}$$

$$=\frac{u^2\sin^2\alpha}{2g}+\frac{1}{4}\frac{2u^2\cos^2\alpha}{g}=\frac{u^2}{2g}.$$

$\therefore$ **Height of the directrix** $=\dfrac{\mathbf{u^2}}{\mathbf{2g}}$. $\quad\cdots\quad$ (22)

In the figure of Art. 9·3, draw OK perpendicular on the directrix and join OS. Then we know $OS=OK$, (from the property of parabola).

$$\therefore\quad \mathbf{OS}=\frac{\mathbf{u^2}}{\mathbf{2g}}. \qquad\cdots\quad (22{\cdot}1)$$

(6) Also, height of the focus $=MS$

$$=\text{height of the vertex}-\tfrac{1}{4}\text{ (latus rectum)}$$

$$=\frac{u^2\sin^2\alpha}{2g}-\frac{1}{4}\frac{2u^2\cos^2\alpha}{g}=-\frac{u^2\cos 2\alpha}{2g}.$$

Since $OM=\frac{1}{2}$ (horizontal range) $=\dfrac{u^2\sin 2\alpha}{2g}$, therefore,

co-ordinates of the focus S are

$$\left(\frac{u^2\sin 2\alpha}{2g},\ -\frac{u^2\cos 2\alpha}{2g}\right). \qquad\cdots\quad (23)$$

Thus, the focus lies above, on or below the horizontal line OX according as $-u^2\cos 2\alpha/2g$ is positive, zero, or negative, *i.e.*, according as $\alpha >=$ or $< \frac{1}{4}\pi$.

In the figure of Art. 9·3, let OQ be the tangent at O. Since the tangent at any point O, of a parabola bisects the angle between OS and the perpendicular from O on the directrix. Draw OK perpendicular on the directrix.

$\therefore \quad \angle KOQ = \angle QOS = \frac{1}{2}\pi - \alpha.$ Join OS.

$\therefore \quad \angle SOX = \angle QOX - \angle QOS = \alpha - (\frac{1}{2}\pi - \alpha) = 2\alpha - \frac{1}{2}\pi.$

Also $OS = u^2/2g$.

$\therefore$ with reference to O as pole and OX the initial line

Polar co-ordinates of focus S are

$$\left(\mathbf{u^2/2g},\ 2\alpha - \tfrac{1}{2}\pi\right). \qquad \cdots \quad (23·1)$$

Also, if R be the horizontal range, H, the maximum height, then with the help of (17) and (21), (13) can be written as

$$(x - \tfrac{1}{2}R)^2 = -\frac{2u^2 \cos^2\alpha}{g}(y - H).$$

Clearly, the point $(\frac{1}{2}R, H)$ is the vertex of the parabola.

(7) If v be the velocity of the particle at any time t, then

$$v^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = u^2 - 2ugt \sin\alpha + g^2t^2$$

[*from (6) & (7)*]

$$= 2g\left\{\frac{u^2}{2g} - (u \sin\alpha . t - \tfrac{1}{2}gt^2)\right\}$$

$$= 2g\left(\frac{u^2}{2g} - y\right) \qquad \cdots \quad (24) \qquad [\textit{from (11)}]$$

$$= u^2 - 2gy. \qquad \cdots \quad (25)$$

From (25), if v be the velocity at any height h, then

$$\mathbf{v^2 = u^2 - 2gh.} \qquad \cdots \quad (26)$$

Thus, from (26) it follows that the magnitude of the velocity at any height h of the path of a projectile is the

same as the velocity acquired by a particle projected vertically upwards with the same inital velocity u.

Again, from (24), we have by using (22),

$$v^2 = 2g \text{ (depth below the directrix).} \qquad \cdots \quad (27)$$

So, the magnitude of the velocity is the same as that acquired by a particle allowed to fall from the directrix.

(8) From the figure of Art. 8'8, OP is an inclined plane, and if $OP = r$, the co-ordinates of P are $(r \cos \beta,\ r \sin \beta)$. Then, since P lies on the projectile, therefore substituting $r \cos \beta$, $r \sin \beta$ respectively for x, y in the equation (12), we have

$$r \sin \beta = r \cos \beta \tan \alpha - \tfrac{1}{2} g \frac{r^2}{u^2} \frac{\cos^2 \beta}{\cos^2 \alpha}.$$

∴ **the oblique range, r**

$$= \frac{2u^2 \cos^2 \alpha}{g \cos^2 \beta} (\cos \beta \tan \alpha - \sin \beta)$$

$$= \frac{2u^2 \cos \alpha \sin (\alpha - \beta)}{g \cos^2 \beta}$$

$$= \frac{u^2}{g \cos^2 \beta} \{\sin (2\alpha - \beta) - \sin \beta\}. \qquad \cdots \quad (28)$$

Cor. From (28), with a given velocity of projection, the maximum range on an inclined plane is obtained when $\sin (2\alpha - \beta)$ is greatest, *i.e.*, when $\sin (2\alpha - \beta) = 1$.

∴ **Maximum oblique range**

$$= \frac{u^2 (1 - \sin \beta)}{g \cos^2 \beta} = \frac{u^2}{g (1 + \sin \beta)}. \qquad \cdots \quad (29)$$

Note 1. The range *down* the plane is obtained by putting $-\beta$ for β in the expression (28).

Note 2. If $\alpha = 0$, *i.e.*, if the particle is projected horizontally, then it follows from (12), that the equation of the path is $y = -\frac{1}{2} g \frac{x^2}{u^2}$, which is obviously a parabola. The independent proof of this is given in Ex. 4 of Illustrative Examples.

9·4. Equations of motion.

It is given in Newton's Second Law of motion that if a force P acting on a particle of mass m in a given direction produces acceleration f (which is in the same direction) then

$$P = mf.$$

From this we have analytically the following equations of motion of a particle *in a given direction* [See § *6·14*]

$$P = m\frac{d^2x}{dt^2} = m\frac{dv}{dt} = mv\frac{dv}{dx}$$

x being the distance of the particle from a fixed origin in the given direction at time t and v the velocity at that instant.

The following equations are for *motion in a plane*

$$X = m\frac{d^2x}{dt^2}, \quad Y = m\frac{d^2y}{dt^2},$$

where X and Y are forces acting on the mass m along the x-axis and y-axis respectively.

For motion in a straight line, when the force is some function say $F(x)$ of the distance x from a fixed point on the line, the equation of motion takes the form

$$m\frac{d^2x}{dt^2} = F(x) \qquad \cdots \qquad \cdots \quad \text{(i)}$$

or, $$mv\frac{dv}{dx} = F(x). \qquad \cdots \qquad \cdots \quad \text{(ii)}$$

(i) may be solved by multiplying both sides by $2\frac{dx}{dt}$ and integrating with respect to t, and we obtain

$$m\left(\frac{dx}{dt}\right)^2 = 2\int F(x)\,dx + C. \qquad \cdots \quad \text{(iii)}$$

(iii) may be solved by integrating both sides with respet to x, and we obtain

$$\tfrac{1}{2}mv^2 = \int F(x)\,dx + C. \qquad \cdots \quad \text{(iv)}$$

(iii) and (iv) are the same.

The integration constant C in (iii) or (iv) may be determined from initial conditions.

9·5. Illustrative Examples.

Ex. 1. *A particle moves on a plane in such a way that its velocity components parallel to the axes of x and y at any instant are $a+by$ and $c+ex$ respectively, where a, b, c and e are constants. Show that the path traced out by the particle is a conic section.*

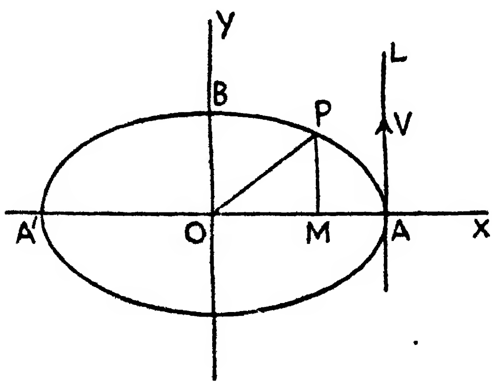
Here the equations of motion are*

$$\dot{x}=a+by,\ \dot{y}=c+ex. \qquad \therefore\ \frac{dy}{dx}=\frac{\dot{y}}{\dot{x}}=\frac{c+ex}{a+by}.$$

Hence, $(c+ex)\,dx-(a+by)\,dy=0.$

Integrating therefore, $cx+\frac{1}{2}ex^2-ay-\frac{1}{2}by^2=A$
which represents the path. It is evidently a conic section, since it is of the second degree in x and y.

Ex. 2. *A particle moves on a plane with an acceleration which is always directed towards a fixed point on it, and varies directly as the distance from it. If the particle be projected from any point in a direction perpendicular to the line joining it to the centre of force, determine its path.*



* Usually in Dynamics, dots are used to denote differentiations with respect to t.

Let a particle move with an acceleration which is always directed towards a fixed point O, and is (say) μ times its distance from O.

Let the particle be projected from a point A with a velocity V along AL, perpendicular to OA.

Choose O as origin, and OAX as x axis, and let the y-axis OY be perpendicular to OX.

P being the position of the particle at any time t, let (x, y) denote the co-ordinates of P.

At P, the acceleration of the particle is $\mu.PO$ along PO, and this is clearly equivalent to the components $\mu.PM$ and $\mu.MO$ parallel to the axes; *i.e.*, the acceleration components are $-\mu x$ and $-\mu y$ parallel to OX and OY.

Hence, the equations of motion are

$$\ddot{x} = -\mu x \qquad \cdots \qquad \cdots \quad \text{(i)}$$

$$\ddot{y} = -\mu y. \qquad \cdots \qquad \cdots \quad \text{(ii)}$$

The general solution of (i), as in § 17·8, is

$$x = A \cos(\sqrt{\mu}t + \epsilon)$$

whence $\quad \dot{x} = -A\sqrt{\mu}\sin(\sqrt{\mu}t + \epsilon).$

At A, when $t=0$, $x=OA=a$ (say), $\dot{x}$, the x-component of velocity, is zero. Hence from above

$$a = A\cos\epsilon, \qquad 0 = -A\sqrt{\mu}\sin\epsilon.$$

$\therefore \qquad \epsilon = 0, \qquad A = a.$

Thus, $\qquad x = a\cos\sqrt{\mu}\,t, \qquad \cdots \qquad \cdots \quad \text{(iii)}$

The general solution of (ii) is $y = A'\cos(\sqrt{\mu}\,t + \epsilon')$,

and so $\quad \dot{y} = -A'\sqrt{\mu}\sin(\sqrt{\mu}\,t + \epsilon').$

As at A, when $t=0$, $y=0$ and $\dot{y}=V$, we get

$$0 = A'\cos\epsilon', \qquad V = -A'\sqrt{\mu}\sin\epsilon',$$

whence $\quad \epsilon' = \frac{1}{2}\pi, \qquad A' = -\frac{V}{\sqrt{\mu}}.$

$\therefore \qquad y = -\frac{V}{\sqrt{\mu}}\cos(\sqrt{\mu}\,t + \tfrac{1}{2}\pi) = \frac{V}{\sqrt{\mu}}\sin(\sqrt{\mu}\,t). \quad \cdots \quad \text{(iv)}$

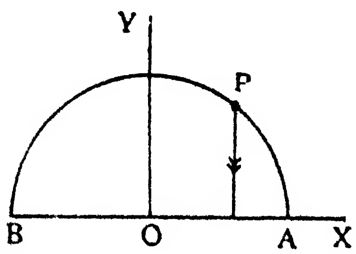
From (iii) and (iv), eliminating t,

$$\frac{x^2}{a^2}+\frac{y^2}{V^2/\mu}=1 \quad \cdots \quad \cdots \quad \text{(v)}$$

giving the path which is evidently an ellipse with axes along OX, OY.

Ex. 3. *A particle moves freely in a semi-circle under a force directed always perpendicularly towards the bounding diameter. Show that the force varies inversely as the cube of the ordinate to the diameter.*

[*C. H. 1942*]



Taking the centre as origin, and the bounding diameter as the x-axis, if (x, y) be the co-ordinates of the particle at any time t, and F denotes the force acting on it per unit mass, directed perpendicularly towards the x-axis, the equations of motion of the particle are

$$\ddot{x}=0 \quad \cdots \quad \cdots \quad \text{(i)}$$

$$\ddot{y}=-F. \quad \cdots \quad \cdots \quad \text{(ii)}$$

Also, from the equation to the circle,

$$x^2+y^2=a^2. \quad \cdots \quad \cdots \quad \text{(iii)}$$

From (i), $\quad \dot{x}=\text{constant}=u$ say.

Also from (iii), differentiating with respect to t,

$$x\dot{x}+y\dot{y}=0, \quad \therefore \quad \dot{y}=-\frac{ux}{y}. \quad \cdots \quad \cdots \quad \text{(iv)}$$

Hence, $\quad \ddot{y}=-u\cdot\dfrac{\dot{x}y-\dot{y}x}{y^2}=-u\cdot\dfrac{uy+u\dfrac{x^2}{y}}{y^2}$

$$=-u^2\frac{x^2+y^2}{y^3}=-\frac{u^2a^2}{y^3}. \qquad [\textit{ from (iii) }]$$

Thus, from (ii), $\quad F=\dfrac{u^2a^2}{y^3} \quad$ *i.e.*, $\quad \propto\dfrac{1}{y^3}$.

Ex. 4. *A particle is projected* **horizontally** *from a point on the top of a tower with a velocity u ; show that the path described is a parabola.*

Let us take the point on the top of the tower as origin and a horizontal line through this as x-axis, and a vertical through this as y-axis, the downward direction of the vertical being considered as positive [See *Fig.* § *8·6*]

Let m be the mass of the particle. Then the equations of motion are

$$m\frac{d^2x}{dt^2}=0 \quad \cdots (1), \qquad m\frac{d^2y}{dt^2}=mg. \quad \cdots (2)$$

Hence, $$\frac{d^2x}{dt^2}=0 \quad \cdots (3), \qquad \frac{d^2y}{dt^2}=g. \quad \cdots (4)$$

Integrating, $$\frac{dx}{dt}=A \quad \cdots (5), \qquad \frac{dy}{dt}=gt+B. \cdots (6)$$

Initially, *i.e.*, at the time $t=0$, horizontal component of velocity is u and vertical component of velocity is zero.

$\therefore$ from (5) and (6), we get $A=u$, $B=0$.

$$\therefore \qquad \frac{dx}{dt}=u \quad \cdots (7) \qquad \frac{dy}{dt}=gt. \quad \cdots (8)$$

Integrating (7) and (8), and noting that when $t=0$, $x=y=0$, we get

$$x=ut \quad \cdots (9)$$

$$y=\tfrac{1}{2}gt^2. \quad \cdots (10)$$

From (9), $t=\dfrac{x}{u}$, and hence from (10) $y=\dfrac{1}{2}g\dfrac{x^2}{u^2}$, or $x^2=\dfrac{2u^2}{g}y$, which is the required path, and which is obviously a *parabola* whose latus rectum is $\dfrac{2u^2}{g}$.

Ex. 5. *A shot is fired with velocity u at a vertical wall whose distance from the point of projection is x. Prove that the greatest height above the level of the point of projection at which the bullet can hit the wall is*

$$\frac{u^4-g^2x^2}{2gu^2}.$$

At what angle is the shot fired in this case ?

Let us take the horizontal and vertical line through the point of projection as the axes of x and y respectively, the positive direction of the x-axis being directed towards the wall and the positive direction of y-axis is vertically upwards. Then we know that the equation of the path is

$$y = x \tan \alpha - \frac{gx^2}{2u^2 \cos^2 \alpha}. \quad \cdots \quad (1)$$

Now, x being the distance of the wall from the point of projection we are to find suitable value of α for which y will be maximum (x remaining constant)

Differentiating (1), $\dfrac{dy}{d\alpha} = x \sec^2\alpha - \dfrac{gx^2}{u^2} \sec^2\alpha \tan \alpha.$ $\quad \cdots \quad (2)$

Hence, $\dfrac{dy}{dx} = 0$ if $\tan \alpha = \dfrac{u^2}{gx}$. It can be easily shown that for this value of α, $\dfrac{d^2y}{d\alpha^2} < 0$. Therefore, the maximum value of y is obtained when $\tan \alpha = \dfrac{u^2}{gx}$, $\cdots$ (3) and the maximum value of y

$$= \frac{u^2}{g} - \frac{gx^2}{2u^2}\left(1 + \frac{u^4}{g^2x^2}\right) = \frac{u^2}{g} - \frac{gx^2}{2u^2} - \frac{u^2}{2g}$$

$$= \frac{u^2}{2g} - \frac{gx^2}{2u^2} = \frac{u^4 - g^2x^2}{2gu^2}.$$

Also from (3), the angle of projection is $\tan^{-1} (u^2/gx)$.

Note. See Example 27 on Chap. VIII.

Examples on Chapter IX

1. If the equation of a plane curve is given by $x = a(\cos \psi + \psi \sin \psi)$, $y = a(\sin \psi - \psi \cos \psi)$, and if ψ increases at a uniform rate k, find the velocity of the point and its inclination to y-axis.

2. A particle describes a parabola under a force which is always directed perpendicularly towards its axis. Prove that the force must be inversely proportional to the cube of the ordinate.

3. A particle describes the catenary $y = c \cosh \frac{x}{c}$ under a force which is always parallel to the positive direction of the y-axis. Find the law of force.

4. If a particle describes a rectangular hyperbola under a force which is always parallel to an asymptote, prove that the force varies as the cube of the distance from the other asymptote.

5. Find the law of force parallel to the axis of y, under the action of which a particle will describe the curve

$$x^2 + 2y^2 = 3x.$$

6. If a particle moves on a plane under a force having components X, Y per unit mass parallel to rectangular axes, (X, Y being functions of x, y in any position), show that the differential equation of its path is

$$\frac{d}{dx}\left[\left(Y - X\frac{dy}{dx}\right) \Big/ \frac{d^2y}{dx^2}\right] = 2X.$$

7. A point moves so that its component velocities parallel to fixed rectangular axes are $a \cos kt$, $b \sin kt$, where a, b and k are constants. Show that the path is an ellipse. Show also that if $a = b$, the acceleration is constant and the path is then a circle.

8. A point moves so that its component velocities parallel to fixed rectangular axes are each constant. Show that the path is a straight line. Prove that the same is true if the component velocities are in a constant ratio.

9. A particle moves with accelerations 2 ft. per sec^2 and 6 feet per sec^2 along the axes of x and y respectively. Initially the particle is at the origin and moving with velocity 2 ft./sec. along the x-axis. Show that the path is a parabola.

10. The position of a point, moving on a plane curve, at time t is given by $a \cos^3 bt$, $a \sin^3 bt$. Show that the point $(\frac{1}{4}a \cos bt, \frac{1}{4}a \sin bt)$ lies on the tangent to the path at that point.

11. (i) A particle moves on a plane, being always acted on by a force, the magnitude of which is proportional to the distance from a fixed point on the plane. Show that the path traced out by the particle is an ellipse if the force is attractive, and a hyperbola if the force is repulsive.

(ii) A particle moves on a plane under a force which is always directed perpendicularly towards a fixed straight line on the plane, and is inversely proportional to the cube of the distance from it. Prove that the path traced out by the particle is a conic-section.

12. Show that the path of a particle moving under a constant force is a parabola or a straight line.

13. Prove that the locus of the foci of all trajectories passing through two given points is a hyperbola.

14. Particles are projected at the same time from the same point in the same vertical plane with different velocities and in different directions.

(i) If the time of flight of each be the same, show that the foci of the different parabolic paths lie on a parabola.

(ii) If the latus rectum of each parabolic path be the same, show that the vertices of the different parabolic paths lie on a parabola.

15. A particle is projected from a given point O in a vertical plane with given velocity u of which the upward vertical component is v. Show that at time $\frac{u^2}{vg}$, its direction of motion is perpendicular to its direction of projection.

16. (i) Three particles P_1, P_2, P_3 are projected in a vertical plane from the same point and at the same time with velocities v_1, v_2, v_3 at angles $\theta_1, \theta_2, \theta_3$ with the horizontal. Show that P_1, P_2, P_3 will always be collinear if

$$\Sigma v_2 v_3 \sin(\theta_2 - \theta_3) = 0.$$

(ii) Three particles P_1, P_2, P_3 are projected in a vertical plane from the same point with velocities v_1, v_2, v_3 at

elevations a_1, a_2, a_3. Show that the foci of their paths will be collinear if

$$\Sigma \sin 2(a_2 - a_3).v_1^{-2} = 0.$$

17. Particles are projected simultaneously with velocities v from a given point in different directions in the same vertical plane. Prove that the centre of the circle on which they lie after t seconds, descends with uniform acceleration.

18. For a maximum range of the projectile, show that the focus of the parabolic path lies on the range.

19. If u be the least velocity with which a projectile can be thrown so as to reach a point k feet vertically high and h feet horizontally distant from the thrower, then show that the angle of elevation of the direction of projection is given by $\tan^{-1}\frac{u^2}{gh}$.

20. Prove that the envelope of the paths of projectiles in vacuo from the same point with the same velocity in the same vertical plane is a parabola with the point of projection as focus.

ANSWERS

1. $ak\phi$, $\frac{1}{2}\pi - \phi$. 3. The force $\propto y$. 5. $-\frac{k}{(3x - x^2)^{3/2}}$.

CHAPTER X

WORK, POWER AND ENERGY

10·1. Work.

Work done by a force acting at a point of a body for any time is the product of the force, and the displacement of the point of application of the force during that time in its own direction.

Fig. (i)

Fig. (ii)

Let a force P be acting on a body at A in the direction AX for any time, and let A move to B during the interval. If AB be in the direction AX, as in the first figure, the work done $= P.AB$, and is *positive*. If the displacement AB of A is in a direction opposite to the direction of P, as in the second figure, the displacement measured in the direction of P is $-AB$, and the work done by the force here is $-P.AB$, which is *negative*.

If the displacement AB be in a direction different from the direction of the force, say making an angle θ with AX as in the third figure, the displacement measured in the direction of P is $AN = AB \cos\theta$, and in this case we get more generally,

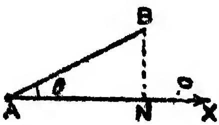


Fig. (iii)

Work done by $P =$ **P.AB cos θ = AB.P cos θ**

$=$ *Force* $\times$ *component of displacement of its point of application along the line of action of the force*

$=$ *Total displacement* $\times$ *component of the acting force along the direction of displacement.*

Note. Evidently the work done is positive if θ be acute, and negative if θ be obtuse. In particular, if $\theta = 90°$, the work done is zero, *i.e.*, no work is done by a force if the resultant displacement of its point of application is perpendicular to the line of action of the force.

If the displacement or its component is in a direction opposite to that of the acting force, work is said to be *done against* the force.

Analytically

If the particle moves along a straight line which we may take as the x-axis from the point x_1 to the point x_2 in time t, expression for the work done in time t can be written analytically as

$$\int_{x_1}^{x_2} F\,dx,$$

where F is the component (constant or variable) of the force acting upon the particle along the x-axis in any position.

In the most general case, if a particle describes a smooth cvrve under any force and if F be the component of the force along the tangent to the path at any instant and ds be the element of the length along the path described in an infinitesimal time dt, then during any interval of time from t_1 to t_2, the corresponding space described being s_1 to s_2, the total work done is

$$\int_{s_1}^{s_2} F\,ds = \int_{t_1}^{t_2} F.v\,dt$$

whether F be constant or variable.

In case of the motion of a particle along a curve if (x, y) be co-ordinates of the position of the particle referred to rectangular axes at any instant and X, Y be the components (constant or vaiable) of the resultant force acting upon the particle at that instant, then as the particle moves from the point (x_1, y_1) to the point (x_2, y_2) on its path, the total work done is

$$\int_{(x_1, y_1)}^{(x_2, y_2)} (X\,dx + Y\,dy).$$

Where X and Y are not constants, they are usually known functions of (x, y).

10·2. Units for measurement of Work.

When a force of one poundal acting on a body displaces its point of application through one foot in its own direction, the amount of work done is defined to be a **Foot-poundal.** This is the British absolute unit of work.

When a force equal to the weight of one pound displaces the point of application through one foot in its own direction, the work done is defined to be one **Foot-pound.** For instance, when a man raises a mass of one pound vertically upwards through one foot, he does work of one foot-pound against the force of gravity, whereas the work done by the weight of the body in this case is negative, and $= -1$ ft.-lb.

As 1 lb. wt. $= g$ poundals, it is clear that

1 ft.-lb. $= g$ foot-poundals.

When a force of one dyne acting on a body displaces its point of application through one centimetre in its own direction, the amount of work done is called an **erg.** This is the c.g.s. absolute unit of work.

As this is very small, a bigger unit of c.g.s. system is

one Joule $= 10^7$ ergs.

As one poundal $= 13800$ dynes roughly,

1 foot-poundal $= 30{\cdot}48 \times 13800$ ergs

$= 420624$ ergs approximately,

and 1 ft.-lb. $= \dfrac{32 \times 420624}{10^7}$ *i.e.*, 1·346 Joules nearly.

10·3. Power.

When an agent (say a man, or a machine or an engine) *is doing work continuously, the rate at which it does work per unit of time is defined to be its power.*

BRITISH UNIT—*When an agent is doing work at the rate of 550 foot-pounds per second, it is said to have one* **Horse-power** (briefly 1 H.P.).

C.G.S. UNIT—*When an agent does work at the rate of 1 Joule (10^7 ergs) per second, its power is said to be one* **Watt.**

We can show easily that 1 H.P. = 746 Watts nearly.

From definition, it follows that

$$Power = Force \times Velocity.$$

10·4. Energy.

Energy *of a body is its capacity for doing work.*

There are two kinds of energy that a body may possess, namely, Kinetic and Potential.

A moving body, by virtue of its motion, possesses a certain capacity for doing work. For, if a force be applied to stop it, it does not stop immediately, but moves a certain distance against the force before it stops. Consequently it does a certain amount of work against the force before coming to rest, and hence at the initial moving state it had in it a capacity for doing this amount of work, *i.e.*, it possessed an energy. If the opposing force be greater or less, the distance moved by the body before coming to rest will be less or greater, and it will be seen below that the amount of the work which that body will perform is definite.

Again, for a body acted on by a given system of forces we may contemplate a suitable position as the standard position. If the body be displaced from this position to some other position, in general a certain amount of work will have to be done against the acting forces. If the body be allowed to go back to the former standard position, the acting forces will do in their turn the above amount of work. The capacity for doing this amount of work then was stored up in the body in its displaced position, which becomes manifest as the body is allowed to go back to its standard position. Thus, a body may possess energy due to its posi-

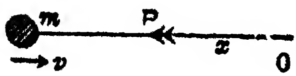
tion. We then formally define the two kinds of energy as follows :

Kinetic Energy *is the capacity for doing work, which a moving body possesses by virtue of its motion, and is measured by the work which the body can do against any force applied to stop it, before its velocity is destroyed.*

Potential Energy *of a body is the capacity for doing work, which it possesses by virtue of its position or configuration, and is measured by the amount of work which the system of forces acting on the body can do in bringing the body from its present position to some standard position.*

10·5. *The kinetic energy of a body of mass m moving with a velocity v is $\frac{1}{2}mv^2$ (in absolute units).*

Imagine a force P to be applied against the direction of motion of the body of mass m moving with a velocity v. Let x be the distance advanced by the body before its velocity is destroyed. Then, since the opposing acceleration produced by the force is P/m, we have



$$0 = v^2 - 2(P/m)x, \text{ whence, } Px = \tfrac{1}{2}mv^2.$$

Thus, the work done by the body against the force before it comes to rest is $\frac{1}{2}mv^2$, and this is, by definition, the measure of the kinetic energy of the body.

It may be noted that the K.E. ultimately depends on m and v, but not on P.

Note 1. It is seen from above that the *unit of energy* is the same as that of work in absolute units (for which $P = mf$ holds) and is usually in *foot-poundals* or *ergs*.

The term **Vis Viva** is used to denote *twice the Kinetic* ...dy, so that Vis Viva $= mv^2$.

10·6. The Principle of Energy.

The change in the kinetic energy of a body is equal to the work done by the acting force.

Let a force P act on a body of mass m for any time, and let u be the initial velocity and v the velocity at the end of the interval, along the line of action of the force. Let x be the displacement of the body in that direction during the interval. The acceleration produced is $\frac{P}{m}$, and so

$$v^2 = u^2 + 2\frac{P}{m}\cdot x.$$

Hence, $\frac{1}{2}\mathrm{mv}^2 - \frac{1}{2}\mathrm{mu}^2 = \mathrm{Px}.$

Now $\frac{1}{2}mv^2$ and $\frac{1}{2}mu^2$ are respectively the final and initial kinetic energy of the body and Px represents the work done by the acting force. Hence, the required result is proved.

Analytically, from the equation of motion

$$P = mv\frac{dv}{dx},$$

integrating w. r. to x, between the limits x_1 to x_2, if v_1 and v_2 be the velocities at those points,

$$\int_{v_1}^{v_2} mv\,dv = \int_{x_1}^{x_2} P\,dx$$

i.e., $\frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2 = P(x_2 - x_1),$

or, $\frac{1}{2}mv^2 - \frac{1}{2}mu^2 = Px$ as above.

Note. The above result which is sometimes spoken of as "*Energy equation*" may also be put in the form

$$\frac{\frac{1}{2}mv^2 - \frac{1}{2}mu^2}{x} = P,$$

which may be expressed as follows:

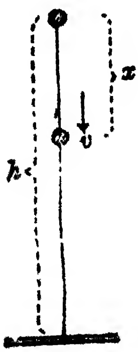
The change in kinetic energy per unit space is equal to the acting force.

10·7. *The potential energy of a body of mass m at a height h above the earth's surface is mgh,* gravity being the only acting force, and earth's surface being taken as the standard position.

For here the force acting on the body is its weight mg vertically downwards, and in bringing the body from its position at a height h to the standard position, namely the earth's surface, the downwards vertical displacement is h, and so the work done by the force acting on the body, which by definition, measures the potential energy of the body at the height h, is mgh.

10·8. Theorem I.

A particle of mass m is allowed to fall from rest at any height h above the ground ; to show that throughout its motion, the sum of its kinetic and potential energies is constant.



Let v be the velocity acquired by the particle at any instant, when it has fallen through a vertical distance x from its starting position. Since the initial velocity is zero, and the acceleration due to gravity is g, we have

$$v^2 = 2gx.$$

Hence, the kinetic energy of the particle

$$= \tfrac{1}{2}mv^2 = \tfrac{1}{2}m.2gx = mgx.$$

Also at this point, the vertical height above the ground being $h - x$,

the potential energy of the particle

$$= mg(h - x).$$

$$\therefore \quad \text{K.E.} + \text{P.E.} = mgx + mg(h - x)$$
$$= mgh,$$

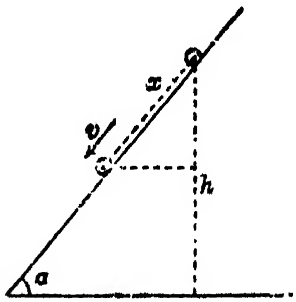
a constant independent of x, and so the same throughout the motion of the particle.

It may be noted that at start the K.E. is zero and the energy is wholly potential and $= mgh$. Again, when it is

just on the point of meeting the ground, the height being zero above the earth's surface, the P.E. is zero, and the energy is wholly kinetic. During the motion of the particle there has been a gradual transformation of energy from potential to kinetic, but the sum-total has remained constant.

Theorem II. *A particle is allowed to slide down a smooth inclined plane; to show that the sum of its kinetic and potential energies is always constant throughout its motion.*

Let a particle of mass m be allowed to slide down a smooth inclined plane of inclination a to the horizon, starting from rest at a point whose height above the ground is h.



The P.E. at the point is then mgh, and the K.E. is zero, so that the total energy at start is mgh.

Let v be the velocity acquired at any instant when the particle has described a distance x along the plane. Since the acceleration down the plane is $g \sin a$.

$$v^2 = 2g \sin a . x.$$

$$\therefore \quad \text{K.E.} = \tfrac{1}{2}mv^2 = mgx \sin a.$$

Now, $x \sin a$ being evidently the vertical height descended by the particle, its height above the ground in this position is $h - x \sin a$, and so

$$\text{P.E.} = mg(h - x \sin a).$$

$$\therefore \quad \text{K.E.} + \text{P.E} = mgx \sin a + mg(h - x \sin a)$$
$$= mgh$$

which is constant and = the initial total energy.

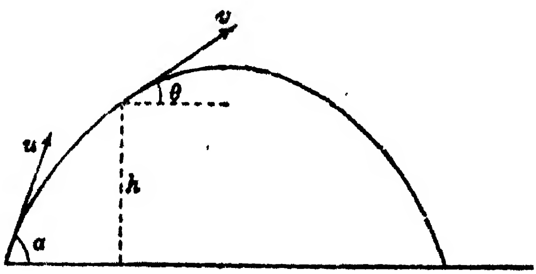
Note. The same result holds if the particle be projected up the plane with any velocity. This is left as an exercise to the student.

Theorem III. *To prove that for a projectile the sum of the kinetic and potential energies is constant throughout its motion.*

Let a particle of mass m be projected from the ground with a velocity u at an angle α to the horizon.

Its initial K.E. is then $\frac{1}{2}mu^2$ and P.E. is zero, so that the total energy at start is $\frac{1}{2}mu^2$.

Let v be the velocity of the projectile at an angle θ with the horizon, when it is at any vertical height h above the ground.



Since there is no horizontal acceleration of the projectile, its horizontal component of velocity remains unchanged and so

$$v \cos \theta = u \cos \alpha. \qquad \cdots \quad \text{(i)}$$

Again, the acceleration due to gravity being g downwards, considering the motion of the projectile in the vertically upward direction,

$$v^2 \sin^2\theta = u^2 \sin^2\alpha - 2gh. \qquad \cdots \quad \text{(ii)}$$

Squaring (i) and adding to (ii),

$$v^2 = u^2 - 2gh.$$

$$\therefore \quad \text{K.E.} = \tfrac{1}{2}mv^2 = \tfrac{1}{2}m(u^2 - 2gh) = \tfrac{1}{2}mu^2 - mgh.$$

Also, h being the vertical height above the ground here,

$$\text{P.E.} = mgh.$$

$$\therefore \quad \text{K.E.} + \text{P.E.} = \tfrac{1}{2}mu^2 - mgh + mgh = \tfrac{1}{2}mu^2$$
= the initial total energy of the projectile, and is thus the same at all heights.

10·9. The principle of Conservation of Energy.

The results of the above article are only simple examples of a fundamental principle in Dynamics known as the Principle of Conservation of Energy, which may be stated as follows :

If a body or a system of bodies move under a conservative system of forces, the sum of its kinetic and potential energies remains constant.

A force system in Dynamics acting on a body is defined to be *conservative* when the work done by the forces of the system, as the body moves from one position to another, depends only on the initial and final positions of the body, but not on any intermediate position, or on the path by which the motion takes place, nor on the velocity or the direction of motion of the body at any moment.

For instance, the force of gravitation is a conservative force. Electrical or magnetic forces are also conservative forces. On the other hand, the force of friction of a rough surface on which a body may slide is not a conservative force. Thus, a body sliding down a rough inclined plane will not have the sum of kinetic and potential energies constant, but this sum will gradually diminish, as can be mathematically verified. Again, when two bodies come into collision, it will be seen in a latter chapter that the sum-total of the energies of the two bodies will in general diminish. The question arises as to what becomes of this energy in those cases of non-conservative forces. This leads of the formulation of the more general form of the Principle of Conservation of Energy in Science.

Energy has been defined to be the capacity for doing work. Now in addition to the energy of motion or energy of position which we have defined above, a body may possess a capacity for doing work on account of its physical condition.

For example, a gas, when in heated state, possesses a capacity for doing work, and can actually be made to do mechanical work in cooling down. Similarly, an electrified body possesses a capacity for doing work on account of its electrified condition. A body emitting sound is in a vibrating condition, and as such, possesses an energy. Light is also another form of energy. Now, if we take into account all the forms of energies recognised by modern science, we may state the principle of conservation of energy in the most general form as follows :

Energy cannot be created, nor can it be destroyed, but it may be transformed from one form to another. The sum-total of the energies in this universe is constant.

10·9(A). Motion of the centre of mass of a system of particles.

Let $m_1, m_2, \dots m_n$ be the masses of a system of particles and G be their centre of mass. If $v_1, v_2, \dots v_r, \dots v_n$ be the velocities and $f_1, f_2, \dots f_n$ be the accelerations of the system at any time t in any given direction, and V be the velocity and F be the acceleration of their centre of masses in the same direction, then

$$V = \frac{\Sigma mv}{\Sigma m}. \quad \dots \quad (1) \qquad F = \frac{\Sigma mf}{\Sigma m}. \quad \dots \quad (2)$$

Let O be the origin and a line OX in the given direction be taken as x-axis. Let $\bar{x}$ be the distance of their C.M. from O and $x_1, x_2, \dots x_n$ be the distances of the positions of the masses from O. Then we have

$$\bar{x} = \frac{m_1 x_1 + m_2 x_2 + \cdots + m_n x_n}{m_1 + m_2 + \cdots + m_n} \quad \dots \quad (3)$$

Now differentiate both sides successively with respect to t.

$$\therefore \quad \dot{\bar{x}} = \frac{m_1 \dot{x}_1 + m_2 \dot{x}_2 + \cdots + m_n \dot{x}_n}{m_1 + m_2 + \cdots + m_n} \quad \dots \quad (4)$$

$$\text{and} \quad \ddot{\bar{x}} = \frac{m_1 \ddot{x}_1 + m_2 \ddot{x}_2 + \cdots + m_n \ddot{x}_n}{m_1 + m_2 + \cdots + m_n} \quad \dots \quad (5)$$

Now, $\dot{\bar{x}} = V$ and $\dot{x}_1, \dot{x}_2, \ldots \dot{x}_n$ are $v_1, v_2, \ldots v_n$

and $\ddot{\bar{x}} = F$ and $\ddot{x}_1, \ddot{x}_2, \ldots \ddot{x}_n$ are $f_1, f_2 \ldots f_n$.

Hence the result.

Cor. Let M denote the total mass of the system of particles. Then $M = \Sigma m$. Hence from (1), $MV = \Sigma mv$. This result can be interpreted in the following way :

The total momentum of a system of particles in any direction is equal to the momentum in the same direction of a particle whose mass is equal to the total mass of the system and which moves with the velocity of the C.M. of the system.

10·10. Illustrative Examples.

Ex. 1. *A locomotive draws a train weighing 200 tons along a level track at a speed of 40 miles per hour, the resistance due to friction etc. amounting to 10 lbs. wt. per ton. What horse-power is it exerting? Find also the horse-power necessary to draw the train at the same speed up an incline of 1 in 200 the frictional resistance being the same as on level.*

[*C. U. 1933*]

The frictional resistance being 10 lbs. wt. per ton, the total force against which the train moves on the level track is 10×200 lbs. wt.

Now since the train moves with a uniform velocity, the resultant force on the train is zero, and so the force exerted by the locomotive is exactly equal to the resisting force *i.e.*, equal to 2000 lbs. wt., and this force displaces the train at the rate of 40 miles per hour

$$= \tfrac{4}{3} \times 44 \text{ ft. per sec.}$$

Thus, work done per second by the locomotive is $2000 \times \frac{176}{3}$ ft.-lbs. As 1 H.P. produces 550 ft.-lbs. of work per second, the horse-power exerted by the locomotive is

$$2000 \times \frac{176}{3} \times \frac{1}{550} = \frac{640}{3} = 213\tfrac{1}{3} \text{ H.P.}$$

In the second case, the component of the weight of the train down the incline $= 200 \times \frac{1}{200}$ tons wt. $= 2240$ lbs. wt. Hence, the total force including the resistance, against which the train moves is

$2240 + 2000 = 4240$ lbs. wt., and this is also the force exerted by the locomotive when the train moves uniformly. For the same speed as before then, the horse-power necessary is

$$4240 \times \frac{176}{3} \times \frac{1}{550} = 452\frac{4}{15} \text{ H.P.}$$

Ex. 2. *A 20 horse-power motor-lorry weighing 5 tons including load, moves up a hill with a slope of 1 in 20. The road resistance is equivalent to 13 lbs. weight per ton and may be supposed independent of the velocity. Find the maximum steady rate at which the lorry can move up the slope, and the acceleration capable of being developed when it is moving at 6 miles per hour.*

The H. P. of the lorry being 20, the work it can do per sec. is 20×550 ft.-lbs., while using its full power.

The component of the weight down the slope here is $5 \times 2240 \times \frac{1}{20} = 560$ lbs. wt., and the road resistance is 13×5 lbs. wt. Hence, the total force against which the lorry moves is $560 + 65 = 625$ lbs. wt.

While moving at a steady rate, the force exerted being equal to this, the velocity v ft. per second when the lorry is working at full power is given by

$$625 \times v = 20 \times 550,$$

or, $\quad v = \frac{88}{5}$ ft./sec. $= \frac{88}{5} \times \frac{30}{44}$, *i.e.*, 12 miles/hr.

which is thus the maximum steady rate with which the lorry can move up the slope.

Again, 6 miles per hour $= 6 \times \frac{44}{30} = \frac{44}{5}$ ft./sec. and when the lorry moves with this velocity, the force P in lbs. wt. which it can exert by using its full power is given by

$$P \times \frac{44}{5} = 20 \times 550, \quad \text{or,} \quad P = 1250 \text{ lbs. wt.}$$

The resisting force being 625 lbs. wt. the resultant upward force is 625 lbs. wt. $= 625 \times 32$ poundals. Hence, the acceleration developed in this case is

$$\frac{625 \times 32}{5 \times 2240} = 1\frac{11}{14} \text{ ft./sec}^2.$$

Ex. 3. Water, originally at rest in a tank, is being pumped out with a speed of 96 feet per second, through a pipe of diameter $3\frac{1}{2}$ inches. Neglecting any work done in changing the level, calculate the horse-power of the engine, if the efficiency of the pumping machinery be 75%. [One c. ft. of water weighs 62·5 lbs.] [*U. P. 1941*]

By **efficiency** of a machine is meant the ratio of the useful work yielded by a machine to the whole amount of work performed by it (a portion of work, which is wasteful work is being done against frictional resistance etc. between the parts of the machinery).

Now x denoting the horse-power of the engine, $550x$ ft.-lbs. of total work is done by it per second. of which the useful work done in this case is

$$\frac{75}{100} \times 550x \text{ ft.-lb. per second.}$$

The area of the section of delivery pipe here is $\pi \times \left(\frac{7}{48}\right)^2$ sq. feet and as water is issuing through it at 96 ft. per sec, the volume of water coming out per sec. $= \frac{22}{7} \times \left(\frac{7}{48}\right)^2 \times 96$ c. ft. of which the mass is $\frac{22}{7} \times \left(\frac{7}{48}\right)^2 \times 96 \times 62{\cdot}5$ lbs.

As the velocity of this water is 96 ft. per sec., its kinetic energy is $\frac{1}{2} \times \left\{\frac{22}{7} \times \left(\frac{7}{48}\right)^2 \times 96 \times 62{\cdot}5\right\} \times 96^2$ in absolute units. Originally this water started from rest, hence the K.E. was zero.

Now by the *principle of energy*, the change in K.E. = the work done by the engine producing it; thus the useful work done by the engine per sec.

$$= \frac{1}{2} \times \frac{22}{7} \times \left(\frac{7}{48}\right)^2 \times 96 \times 62{\cdot}5 \times 96^2 \text{ foot-poundals}$$

$$= \frac{1}{2} \times \frac{22}{7} \times \left(\frac{7}{48}\right)^2 \times 96 \times 62{\cdot}5 \times 96^2 \times \frac{1}{32} \text{ foot-pounds,}$$

and this as shown above must be equal to

$$\frac{75}{100} \times 550 \times x.$$

Hence,

$$x = \frac{1}{2} + \frac{22}{7} \times \left(\frac{7}{48}\right)^2 \times 96 \times 62{\cdot}5 \times 96^2 \times \frac{1}{32} \times \frac{100}{75} \times \frac{1}{550}$$

$= 140$, which is the H.P. of the engine.

Examples on Chapter X

1. A man weighing 10 stones walks one mile up an incline of 1 in 7. Find the work done. If he takes 20 minutes for the walk, find the H.P. at which he works.

2. A horse pulls a block of stone on a level ground through h yds, with a force 545 lbs. wt. by means of a string inclined at $60°$ to the horizon, and does the same amount of work as done by a pump in raising 38 kilograms of water from a depth of 23 metres. Find h, given 1 foot-poundal $= 419520$ ergs.

3. Find the work done by gravity on a stone having a mass of $\frac{1}{2}$ lb. during the tenth second of its fall from rest.
[*U. P. 1943*]

4. A railway wagon weighing 10 tons is started from rest by a horse, which exerts a constant pull of 120 lbs. wt. The frictional resistances are 9 lbs. weight per ton. How far does the horse move the wagon in one minute, and at what H.P. is the horse working at the end of the minute ?
[*U. P. 1945*]

5. A motor-boat of 40 H.P. working at full speed moves at the rate of 20 miles per hour. What is the resistance of water to its motion ?

6. An engine of 400 H.P. is drawing a train of 200 tons mass, up an incline of 1 in 280, at 30 miles per hour. Find the road resistance in pounds weight per ton,

7. A train whose weight is 100 tons is moving up an inclined plane with a uniform speed of 45 miles per hour, inclination being 1 in 100. Find the horse-power of the engine, the resistance due to friction etc. being $\frac{1}{20}$ of the weight.
[*O. U. 1940*]

8. If the resistance and the friction of the rails be 1 lb. wt. per ton, what is the horse-power of an engine which will maintain a speed of 40 m.p.h. in a train of 80 tons on a level ? What additional horse-power would be required to maintain that speed up an incline of 1 in 200 ?

9. A cyclist can ride down a slope of 1 in 80 without any effort at a steady speed of 10 m.p.h. If the cycle and the rider together weigh 200 lbs., find the horse-power exerted when the cyclist rides at the same speed uphill against the same frictional resistances.

If frictional resistances vary as the square of the speed, find what speed the cyclist can attain on the level if the same horse-power is exerted as before.

10. A locomotive of mass 20 tons pulls a mass of 200 tons from rest with a constant force along a horizontal track such that a speed of 60 m.p.h. is attained in the first 5 miles. Prove that at the end of this journey the locomotive is working at the rate of about 361 H.P.

[*All frictional resistances are to be neglected.*]

11. A train of total mass 200 tons is travelling on the level at a constant rate of 60 m.p.h., the engine working at 400 H.P. If the resistances apart from air-resistance are 2000 lbs. wt., find in lbs. wt. the air resistance.

If the air-resistance varies as the square of the speed and the engine is drawing the same train up a gradient of 1 in 112 at a steady rate of 30 m.p.h., at what horse-power is it working, assuming frictional resistance to be the same as on the level.

12. A rifle bullet loses $\frac{1}{10}$th of its velocity in passing through a wooden board. Find through how many such uniform boards it would pass before being stopped, assuming the resistance of the boards to be uniform. [*C. P. 1937*]

[*Apply the principle of energy.*]

13. Find the horse-power of an engine which can project 10000 lbs. of water per minute with a velocity of 80 ft. per second. [*C. U. 1944*]

14. A fire-engine raises 1200 gallons of water per minute through a height of 6 feet, and discharges with a velocity of 32 feet per second. Find the horse-power of the engine, given that one gallon of water weighs 10 lbs. [*U. P. 1940*]

A 15. Show that the horse-power required to pump 1000 gallons of water per minute from a depth of 50 feet, and deliver it, through a pipe of cross-section 6 sq. inches, is about $34\frac{1}{2}$. [Assume 1 cubic ft. of water $= 6\frac{1}{4}$ gallons, and that 1 gallon of water weighs 10 lbs.]

16. The water-supply of a hill station is provided with pumps of 5000 H.P. which raise the water a distance of 4200 ft. (vertical). The efficiency of the pump is 92·4 per cent. Assuming that the pumps operate continuously, find how many gallons per day are provided for consumption.

[*A gallon of water weighs 10 lbs.*]

A 17. A labourer has to supply bricks to a bricklayer vertically above him, at a height 12 ft. He throws them up so that they reach the bricklayer with a velocity of 12 ft. per sec. What proportion of his work could he save if he threw them so that they might just reach the bricklayer ?

[*U. P. 1944*]

18. An engine draws a train along a level line starting from rest. If the pull of the engine be constant till the steam is shut off, and the resistance F be constant throughout the journey, then the greatest rate of working is

$$\frac{2lF^2t}{Ft^2-2Ml},$$

where M is the mass of the train, l the length of the journey and t the time occupied by it.

19. A train, whose mass including that of the engine is M, is moving along a level track. When the speed of the train is v_1, its acceleration is f_1 and the resistance to motion is R_1. When the speed of the train is v_2, its acceleration is f_2 and the resistance to motion is R_2. If the engine works at a constant rate H, prove that

$$H(v_2-v_1)=v_1v_2(R_1-R_2)+Mv_1v_2(f_1-f_2).$$

20. An engine of weight W tons can exert a maximum tractive effort of P tons weight, and develop at most H horse-power. The resistances to motion are constant and equal to R tons weight. Show that starting from rest, the engine will first develop its full horse-power when its velocity is $\dfrac{55H}{224P}$ ft./sec. after at least $\dfrac{55WH}{224Pg(P-R)}$ seconds. What is the greatest velocity which the engine can attain ?

21. A heavy uniform chain, of length $2l$, hangs over a smooth fixed pulley, the length $l+c$ being at one side and $l-c$ at the other ; if it be released from a state of rest, show by the principle of energy, that the chain will slip off of the pulley in time

$$\left(\frac{l}{g}\right)^{\frac{1}{2}} \log \frac{l+\sqrt{(l^2-c^2)}}{c}. \qquad [\text{ C. H. 1965 }]$$

22. A heavy uniform string of mass M and length $2a$, is placed at rest symmetrically over a smooth peg and has particles of masses m and m' attached to its extremities ; if $m > m'$, show by principle of energy that when the string runs off the peg, its velocity is

$$\left\{\frac{M+2(m-m')}{M+m+m'}ag\right\}^{\frac{1}{2}}. \qquad [\text{ C. H. 1962 }]$$

ANSWERS

1. 105600 ft.-lbs. ; ·16 H.P.
2. $7\frac{4}{9}$.
3. 152 ft.-lbs.
4. $77\frac{1}{7}$ feet ; [illegible] H.P.
5. 750 lbs. wt.
6. 17.
7. $806\frac{2}{3}$ H.P.
8. $8\frac{8}{15}$ H.P. ; [illegible] H.P.
9. $\frac{2}{15}$ H.P. ; $10\sqrt{2}/2$ m.p.h.
11. 500 lbs. wt. ; 490 H.P.
12. $8\frac{8}{11}$.
13. [illegible] H.P.
14. 8 H.P.
16. 5227200.
17. $\frac{4}{15}$.
20. $55H/224R$ ft./sec.

CHAPTER XI

IMPULSIVE FORCES

11·1. Impulse.

The impulse of a force acting on a body for any time is the product of the force and the time during which it acts.

Let P be the force acting on a particle of mass m for any time t.

Then, by definition, impulse I of the force is

$$\mathbf{I = Pt.} \qquad \cdots \qquad \cdots \quad (1)$$

Now let u be the initial velocity of the particle in the direction of the force, and v the velocity at the end of the time t in the same direction. Since the acceleration produced by the force in the mass is P/m, we get

$$v = u + \frac{P}{m} \cdot t.$$

Then, $\qquad Pt = m(v - u) = mv - mu, \qquad \cdots \quad (2)$

i.e., $\qquad \mathbf{I = mv - mu}. \qquad \cdots \qquad \cdots \quad (3)$

Hence, *Impulse = change of momentum (in the direction of the force).*

Note. The relation (2) is sometimes spoken of as "*Momentum equation*". If the force which acts on a body for any time t be variable, we should divide the whole time t into infinitely small portions, each so small that during that small interval the measure of the acting force may be considered as constant, and then find the impulse during each of these small intervals, and finally add them up to get the total impulse. It is evident that during each small interval the impulse is equal to the change of momentum produced in the body, and adding up, the total impulse = the total change of momentum produced.

Analytically, $\quad I = \int_{t_0}^{t} P\,dt = \int_{u}^{v} m\frac{dv}{dt}\,dt = mv - mu.$

11·2. Impulsive forces.

Let a force act on a body of given mass m for any time t. Suppose we know the initial position and motion of the body at the instant when the force begins to act. The effect of the force acting for time t will be in general, to produce a definite displacement, as also to produce a definite change of momentum, and these two being known, we know the final position and motion of the body completely. Now, it has been shown that the change of momentum produced by the force is known if we know the impulse of the force for the time. Thus, the whole effect of a force acting on a body for any given time will be known if we know the impulse of the force during the interval, and the displacement of the body produced during the interval.

Now there are particular types of forces, which are sudden forces in the nature of blows, of extremely short duration, sufficiently large so as to produce in a body a finite change of motion, *i.e.*, a finite change of momentum, though during that short interval for which the force acts, the body has not time enough to have any appreciable displacement. As an example, when a cricket bat hits a ball, the duration of action of the force on the ball is the time for which the ball actually remains in contact with the bat, and this is extremely small. But during the small interval, practically the twinkling of an eye, the motion of the ball is definitely altered. The ball meets the bat and immediately separates from it, and during the period of actual contact the displacement of the ball is negligible. The effect of the hit, therefore, is to produce a sudden change of motion of the ball practically at the same spot where the bat meets the ball. With this newly acquired velocity the ball moves on, but that motion is a subsequent affair when the hit has already done its effect, and is no longer acting. As in the case of such sudden forces the time of action is infinitely small, and the displacement of the body negligible, the acceleration produced by the force cannot be determined in general, and hence the magnitude of the force in ordinary units cannot be determined, nor is required for any purpose. To know the whole effect of the force in such a case, it will be sufficient

to know the change of momentum produced by it, for that would give us the newly acquired velocity of the body, from which the subsequent positions and motion of the body can be studied. The force in this case is very large, but the duration is very small, and the product of these two *i.e.*, the impulse of the force is finite, as is evidenced by the change of momentum produced, which equals this impulse. The effect of such forces depending solely on their impulse, the measure of such forces are also expressed by the impluse as determined by the change of momentum produced. Hence, such sudden forces are termed *impulsive forces.* We give the formal definition as follows :

An **Impulsive force** *is a very large force of an extremely short duration, such that the impulse of the force, that is, the change of momentum poduced by it in a body, is finite but the displacement of the body during the short interval is negligible. The measure of such a force is given by its impulse only, which also gives the whole effect of such a force.*

11·3. Principle of Conservation of linear momentum.

When two (or any number of) bodies move under their mutual action and reaction only (whether finite or impulsive), and no external forces act on the system, the sum-total of their momenta along any direction is constant.

For if A and B be two bodies moving under no external forces but their mutual action and reaction, by Newton's third law of motion, the action of A on B is at every instant equal and opposite to the reaction of B on A; again, so long as there is action, there is also the reaction, and thus the time for which the two forces (action and reaction) act is the same for both. Hence, the impulses of the two forces are equal and opposite, and as the impulse of a force is known to be equal to the change of momentum produced by it, it follows that the change of momentum produced in A is equal and opposite to the change of momentum produced

in B. Hence, taken together, the total change of momenta of A and B is zero, or in other words, the sum-total of the momenta of A and B along any direction is unchanged.

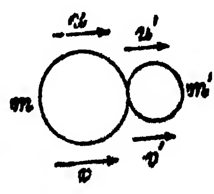
The result then can be extended to the case of any number of bodies moving under mutual actions and reactions only.

Note. If on a system there be external forces acting, of which the algebraic sum of the resolved parts in a *particular direction* is zero, then the sum-total of the momenta of the system is constant in that direction only.

Two well-known examples of the above principle in case of *impulsive action and reaction* between two bodies are given below :

(A) Collision of two bodies.

When two moving bodies A and B come into collision, the time for which they remain in contact is extremely small, but within that small time the velocities of the bodies are definitely altered by the mutual action and reaction between the two bodies. The forces of action and reaction are therefore of the nature of impulsive forces, equal and opposite to one another. The line of action of those forces in case of smooth bodies is clearly along the common normal at their point of contact, and this line is known as the *line of impact.*



Let m and m' be the masses of the two bodies, u and u' their velocities along the line of impact immediately before collision, v and v' their velocities measured in the same direction immediately after collision. Then by the principle of conservation of momentum proved above, we get in this case

$$mv+m'v'=mu+m'u'.$$

This gives us one equation between the two unknowns v and v'. A second equation of this case will be obtained in the next chapter.

Cor. If the two bodies after collision do not separate, but coalesce to form one body, *i.e.*, if the bodies be *inelastic* (for instance, in case of two clay balls coming into collision) the common velocity V of this single body, whose mass is evidently $m+m'$, is found from the principle of conservation of momentum, by the equation

$$(m+m')\,V = mu + m'u'.$$

(B) Motion of a Shot and a Gun.

When a gun is fired, the gunpowder is suddenly converted into gas by explosion, and this gas in trying to expand, forces the shot forwards. An equal and opposite reaction is exerted on the gun. The duration of this expansive force is extremely small, only so long as the shot moves within the barrel. Hence, the force is impulsive in nature. As the shot moves within the barrel, the volume of the enclosed gas gradually expands and so the expansive force is variable, but at every instant the force on the shot and the reaction on the gun are equal and opposite. The total changes of momentum of the shot and the gun are therefore equal and opposite.

Initially, both the shot and the gun were at rest. Hence, when the shot emerges out of the muzzle of the gun, the momentum of the shot forwards is equal to the momentum generated in the gun backwards.

Thus, if m and M be the masses of the shot and the gun, v being the muzzle velocity with which the shot emerges from the gun, the gun will recoil with a velocity V given by

$$MV = mv.$$

11·4. Illustrative Examples.

Ex. 1. *A marble whose mass is 2 ounces is dropped on a horizontal floor from a height of 25 feet and rebounds to a height of 16 feet. Find*

the impulse, and the average force between the marble and the floor if the time during which they are in contact be $\frac{1}{10}$ of a second.

On hitting the floor, the velocity of the marble

$$=\sqrt{2g\times 25}=\sqrt{2.32.25}=8\times 5=40 \text{ ft./sec.}$$

and on leaving it the velocity

$$=\sqrt{2g\times 16}=\sqrt{2.32.16}=8\times 4=32 \text{ ft./sec.}$$

The mass of the ball $=\frac{2}{16}=\frac{1}{8}$ lbs.

$\therefore$ Impulse = change of momentum,

$$=\tfrac{1}{8}\{32-(-40)\}=\tfrac{1}{8}\times 72=9 \text{ units.}$$

If P be the average force between the marble and the floor, the resultant force upwards (taking into account the weight of the body) producing the change of momentum is $P-\frac{1}{8}.32$.

$\therefore \quad (P-\frac{1}{8}.32)\times\frac{1}{20}=9, \quad \text{or,} \quad P-4=180.$

$\therefore \quad P=184$ poundals.

Ex. 2. *How far must a weight of 5 cwt. fall freely to drive a pile weighing 640 lbs., 3 inches into the ground against an average resistance of 5 tons, assuming that the weight moves on with the pile.*

[*C. U. 1944*]

Let h ft. be the height through which the body of mass 5 cwt. $(=5\times 112$ lbs.) falls freely before it hits the pile.

Its velocity then immediately before impact is given by

$$v^2=2gh, \quad \text{or, } v=\sqrt{2gh}=\sqrt{2\times 32\times h}=8\sqrt{h} \text{ ft./sec.}$$

If v' be the velocity of the weight and the pile combined, after impact, then from the principle of conservation of momentum, we get

$$(560+640)v'=560v,$$

or, $$v'=\tfrac{7}{15}\times 8\sqrt{h} \text{ ft./sec.}$$

The average resistance of the ground is

5 tons wt. $=5\times 2240$ lbs. wt. upwards,

whereas the downward weight of the system

$=560+640=1200$ lbs. wt.

Hence, the resultant force acting on the system after impact $=5\times 2240-1200=10000$ lbs. wt. upwards, and against this the system moves 3 inches *i.e.*, $\frac{1}{4}$ ft. before coming to rest.

Hence, the work done by the acting force

$$= -(10000 \times 32 \times \tfrac{1}{4}) \text{ ft.-poundals},$$

which (being in absolute units) = the change in the kinetic energy of the system.

Thus, $\quad 0 - \frac{1}{2} \times 1200 \times (\frac{7}{15} \times 8\sqrt{h})^2 = -(10000 \times 32 \times \frac{1}{4})$.

Hence, $\quad h = \dfrac{10000 \times 8 \times 15^2}{600 \times 7^2 \times 8^2} = \dfrac{1875}{196} = 9\dfrac{111}{196}$ ft.

Ex. 3. *A shell, lying in a straight smooth horizontal tube, suddenly explodes and breaks into portions of masses m and m'. If d is the distance apart of the masses after a time t', show that the work done by the explosion is*

$$\frac{1}{2}\frac{mm'}{m+m'}\frac{d^2}{t^2}.$$

Let v and v' be the velocities after explosion of the protions m and m respectively (in opposite directions) along the tube. Then by the principle of conservation of momentum,

$$mv - m v' = 0, \quad \text{or,} \quad mv = m'v'. \qquad \cdots \quad \text{(i)}$$

Also, the distance apart between the portions after t secs. is,

$$(v + v')\, t = d. \qquad \cdots \qquad \cdots \quad \text{(ii)}$$

$\therefore$ by (i) & (ii), $\quad \dfrac{v}{m'} = \dfrac{v'}{m} = \dfrac{v+v'}{m+m'} = \dfrac{d}{t(m+m')}. \qquad \cdots$ (iii)

Now the work done by the explosion

$$= \text{the kinetic energy generated by it} = \tfrac{1}{2}(mv^2 + m'v'^2)$$

$$= \frac{1}{2}\left[mm'^2 + m'm^2\right] \times \left\{\frac{d}{t(m+m')}\right\}^2 = \frac{1}{2}\frac{mm'}{m+m'}\frac{d^2}{t^2}. \quad [\text{ by (iii) }]$$

Ex. 4. *A mass m after falling freely through a feet begins to raise a mass M greater than itself and connected with it by means of an inextensible string passing over a fixed pulley. Show that M will have returned to its original position at the end of time*

$$\frac{2m}{M-m}\sqrt{\frac{2a}{g}}.$$

Find also what fraction of the visible energy of m is destroyed at the instant when M is jerked into motion.

The velocity acquired by m in falling freely through a distance a is given by

$$v^2=2ga, \text{ or, } v=\sqrt{2ga}.$$

v' denoting the velocity of the system when M is jerked into motion, by the principle of conservation of momentum,

$$(M+m)\,v'=mv=m\sqrt{2ga},$$

or, $\quad v'=\dfrac{m}{(M+m)}\sqrt{2ga}$ which represents the velocity with which the heavier mass M begins to move upwards.

For the subsequent finite motion, the acceleration of the heavier mass M downwards is $\dfrac{M-m}{M+m}g$.

Hence, M will at first rise and subsequently fall, and come back to its original position after a time t given by

$$\frac{m}{(M+m)}\sqrt{2ga}.t-\frac{1}{2}\frac{M-m}{M+m}g.t^2=0,$$

or, $$t=\frac{2m}{M-m}\sqrt{\frac{2a}{g}}.$$

Again, the K.E. of the system immediately before M is jerked into motion is $\frac{1}{2}mv^2=\frac{1}{2}m.2ga=mga$, and immediately after, it is

$$\tfrac{1}{2}(M+m)\,v'^2=\tfrac{1}{2}(M+m)\frac{m^2}{(M+m)^2}\cdot 2ga$$

$$=\frac{m^2}{M+m}\cdot ga.$$

Hence, the fraction of the visible energy destroyed

$$=\left(mga-\frac{m^2ga}{M+m}\right)\Big/mga=\left(1-\frac{m}{M+m}\right)=\frac{M}{M+m}.$$

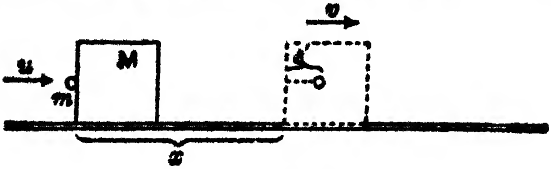
Ex. 5. *A shot whose mass is m penetrates a thickness s of a fixed plate of mass M. If M were free to move, show that the thickness penetrated would be*

$$s\Big/\left(1+\frac{m}{M}\right).$$

Let u denote the initial velocity of the shot, and P denote the force of resistance to penetration. In the first case when the plate is fixed, s denoting the distance moved over by m into the plate before it comes to rest, we have, by the principle of energy,

$$0-\tfrac{1}{2}mu^2=-Ps, \quad\text{or,}\quad \tfrac{1}{2}mu^2=Ps. \qquad \cdots \quad (1)$$

In the second case, when the plate is free to move, as the shot penetrates the plate, its velocity diminishes due to the resisting force P,



and the velocity of the plate increases from zero due to the equal and opposite reaction acting on it. So long as the velocity of the shot remains greater than that of the plate, penetration continues, until, when both acquire a common velocity v (say), there will be no further penetration, and thus P ceases to act. Let x denote the distance moved over by the plate up to this instant, and s' the thickness penetrated by the shot in this case. Then applying the principle of energy in this case, we get,

$$\tfrac{1}{2}(M+m)v^2-\tfrac{1}{2}mu^2=-P(x+s')+Px=-Ps'. \quad \cdots \quad \text{(ii)}$$

Also from the principle of conservation of momentum,

$$(M+m)\,v=mu.$$

Hence, from (ii),

$$\frac{1}{2}\frac{m^2u^2}{M+m}-\tfrac{1}{2}mu^2=-Ps',$$

or, $$\tfrac{1}{2}mu^2\cdot\frac{M}{M+m}=Ps'. \quad \cdots \quad \text{(iii)}$$

From (i) and (iii),

$$\frac{s'}{s}=\frac{M}{M+m}, \quad \text{or,} \quad s'=\frac{sM}{M+m}=s\Big/\left(1+\frac{m}{M}\right).$$

Ex. 6. *A continuous jet of water which is issued from a circular pipe of $3\frac{1}{2}$ inches diameter strikes a wall at right angles with a velocity of 38·4 ft. per sec. and then drops straight down. Find the pressure on the wall.*

[*1 cub. ft. of water weighs* $62\frac{1}{2}$ *lbs.*]

Here we are dealing with a succession of impacts or impulsive forces. The amount of momentum destroyed per second due to the reaction of the wall on the jet gives the average force or thrust on the surface (which is equal and opposite to the reaction).

Area of the cross-section of the pipe $= \pi \left(\dfrac{7}{4 \times 12}\right)^2$ sq. ft.

$\therefore$ the mass of water reaching the wall per sec.

$$= \pi \left(\frac{7}{4 \times 12}\right)^2 \times 38\tfrac{4}{10} \times 62\tfrac{1}{2} \text{ lbs.}$$

and its velocity is $38\frac{4}{10}$ ft. per sec., which is reduced to zero after striking the wall.

$\therefore$ the momentum destroyed per sec.

$$= \frac{22}{7} \times \frac{7^2}{4^2 \times 12^2} \times \frac{192}{5} \times \frac{125}{2} \times \frac{192}{5}$$

$$= 22 \times 7 \times 8 \times 5 \text{ units. (absolute).}$$

$\therefore$ pressure on the wall $= 22 \times 7 \times 8 \times 5$ poundals

$$= \frac{22 \times 7 \times 8 \times 5}{32} \text{ lbs. wt.} = 192{\cdot}5 \text{ lbs. wt.}$$

Ex. 7. *Find the average pressure per square foot on the ground due to a rainfall of* $1\frac{1}{5}$ *inches in* 5 *hours assuming that rain falls freely from a height of* 900 *ft.*

[1 *cubic foot of water weighs* $62\frac{1}{2}$ *lbs.*]

The velocity of rain on striking the ground

$$= \sqrt{2g \times 900} = \sqrt{2 \times 32 \times 900} = 8 \times 30 \text{ ft. per sec.}$$

The volume of rain that falls on a square foot in 5 hours

$$= 1^2 \times \tfrac{6}{5} \times \tfrac{1}{12} \text{ cub. ft.} = \tfrac{1}{10} \text{ cub. ft.}$$

$\therefore$ the mass of rain that falls on a square foot in 5 hours

$$= \tfrac{1}{10} \times \tfrac{125}{2} \text{ lbs.}$$

$\therefore$ momentum destroyed per sec. due to reaction of the ground on the rain drops is

$$\frac{1}{10} \times \frac{125}{2} \times 8 \times 30 \times \frac{1}{5 \times 60 \times 60}.$$

$\therefore$ pressure on the ground per sq. foot (being equal and opposite to the reaction of the ground)

$$= \frac{125 \times 8 \times 30}{10 \times 2 \times 5 \times 60 \times 60} \text{ poundals} = \tfrac{1}{12} \text{ poundals.}$$

Examples on Chapter XI

1. A tennis ball of weight 2 ozs. is dropped from a height of 9 ft. on to a racket which is held still in a horizontal position, and rebounds vertically to a height of 4 ft. Find the impulse on the racket and the average force on the ball if the impact lasted $\frac{1}{12}$th of a second.

2. A 4 oz. cricket ball moving horizontally at 80 ft. per sec. was hit straight back with a speed of 48 ft. per sec. If the contact lasted $\frac{1}{20}$ second, find the average force exerted by the bat.

3. A body of mass 5 lbs. moving with a velocity of 12 ft. per sec. impinges directly on a mass of 10 lbs. moving with a velocity of 6 ft. per sec. in the same direction and adheres to it. Find the velocity of the compound body.

If they were moving in opposite directions before impact, show that after impact they are brought to rest.

4. A railway truck of weight 10 tons moving with a velocity of 8 ft. per sec. impinges on another truck of weight 4 tons at rest, and travels after impact with a velocity of 5 ft. per sec. Find the velocity of the second truck and also the loss of K.E., in ft.-lbs. due to the impact.

5. A boy of mass M standing on perfectly smooth ice picks up a stone of mass m as it is sliding towards him with velocity v. At what rate will boy begin to slide ?

6. A shell, moving horizontally with a velocity of 1600 ft. pe sec., is split into two parts by an internal explosion. The velocity of one part is reduced to 1100 ft. per sec. in the same line. Find the velocity with which the other part moves if its mass is $\frac{1}{5}$th of the whole.

7. A hammer weighing 1 lb., striking a nail weighing 1 oz. with a horizontal velocity of 34 ft, per sec., drives the nail 1 inch into a fixed block of wood. Find the resistance of the wood, assuming that the hammer moves with the nail after the blow.

8. An inelastic mass of 6 cwt. falls freely from a height of 9 ft. upon a pile of mass 12 cwt., and drives it into the ground. If the average resistance of the ground to penetration by the pile be equal to $2\frac{1}{10}$ tons wt., find the distance through which the pile is driven by the blow.

9. A shot of mass 14 lbs. is fired horizontally with a velocity of 1280 ft. per sec. from a gun of mass half-a-ton. If the recoil of the gun be resisted by a constant force of one ton weight, find the distance through which the gun moves back and the time it takes before coming to rest.

10. A gun of mass 1500 lbs. fires a shot of 15 lbs. and recoils $12\frac{1}{2}$ ft. up a smooth inclined plane of 1 in 8. Find the muzzle velocity of the shot.

11. Masses m and $2m$ are connected by a string which passes over a smooth pulley. The ascending body picks up a mass m at the end of 3 seconds. Find the resulting motion. [*C. U. 1943*]

12. Two bodies each of mass 2 lbs. at rest are connected by a string passing over a small smooth fixed pulley ; a lump of putty whose mass is 1 lb. falls on one with a velocity of 10 ft. per sec. and sticks to it. Find the velocity of the system $\frac{1}{2}$ a second after the impact.

13. A body of mass 6 lbs. after falling freely through 4 ft. lifts a body of mass 10 lbs. from rest vertically upwards by means of a light inelastic string passing over a smooth fixed pulley. How far will the 10 lbs. mass rise ? What is the impulsive tension of the string when the body is lifted ?

14. Two particles of masses 10 lbs. and 4 lbs. connected by a light inextensible string passing over a smooth fixed pulley are left free. If the heavier particle reaches the ground (assumed inelastic) after descending a distance of 21 feet, find how many seconds later it will be jerked off the ground and the height to which it will rise subsequently.

[*The portions of the string on either side of the pulley are assumed sufficiently long.*]

15. Assuming that rain falls freely from a height of 729 ft., find the pressure per square foot due to a fall of $\frac{3}{4}$ inches in 2 hours.

[*A cubic foot of water weighs* 1000 *ozs.*]

16. Water flows at a velocity of 4 ft. per sec. from the lower end of a vertical pipe 2·4 inches in diameter and after falling freely 21 ft. strikes a horizontal plane without rebounding. Find the impulsive pressure on the plane in lbs. wt.

17. A continuous jet of water is thrown by a fire-engine so as to strike a wall at right angles with a velocity of 72 ft. per sec. If the section of the hose be 4 square inches and the water rebounds with a velocity of 24 ft. per sec., find the pressure on the wall.

18. An inelastic ball of mass 40 lbs. is dropped from a height of 96 ft. above the ground and at the same time a second ball of mass 20 lbs. is thrown vertically upwards to meet the former. In order that immediately after collision the balls may be at rest, show that the second ball must be projected with a velocity of 96 ft. per sec.

19. Two masses m_1 and m_2 moving along the same straight line collide and form one body. If a force applied to stop the masses individually before collision would bring them to rest in distances x_1 and x_2 respectively, find the distance in which the joint body after collision would be brought to rest by the same force.

20. A shell is dropping vertically, and when its velocity is v and height h, it bursts into two fragments of masses m_1 and m_2, which, after describing parabolic orbits, reach the ground in t_1 and t_2 seconds. Show that

$$\frac{m_1}{m_2} = \frac{t_1\,(vt_2 + \frac{1}{2}at_2^2 - h)}{t_2\,(h - vt_1 - \frac{1}{2}gt_1^2)}.$$

21. A shell fired from a gun explodes into two equal parts when at the highest point of its path. If one of the parts falls vertically from rest, shew that the other will describe a parabola of which the latus rectum will be four times that of the original parabola.

22. A gun is mounted on a gun carriage movable on a smooth horizontal ground, and the gun is elevated at an angle a to the horizon ; a shot is fired and leaves the gun in a direction inclined at an angle θ to the horizon ; if the mass of the gun and its carriage be n times that of the shot, show that

$$\tan\theta = \left(1 + \frac{1}{n}\right)\tan a.$$

23. A shot of mass m is fired with a velocity u relative to a gun mounted on a carriage which is free to move on a smooth horizontal ground, the gun being elevated at an angle a to the horizon. If the mass of the gun and carriage be M, find the range of the shot on the ground.

24. A gun of mass M fires a shell of mass m horizontally and the energy of explosion is such as would be sufficient to project the shell vertically to a height h. Show that the velocity of recoil of the gun is

$$\left\{\frac{2m^2gh}{M(m+M)}\right\}^{\frac{1}{2}}.$$

25. A bullet of mass m, moving with velocity u, strikes a block of mass M, which is free to move in the direction of the motion of the bullet and is embedded in it. Show that the loss of kinetic energy is

$$\frac{1}{2}\,\frac{mM}{m+M}u^2.$$

26. A body moving along a straight line, splits into two parts of masses m_1 and m_2 by an internal explosion which generates kinetic energy E. Show that if after explosion the parts move in the same line as before, their relative speed is

$$\sqrt{2E\left(\frac{1}{m_1}+\frac{1}{m_2}\right)}.$$

27. A bullet of mass m is fired with a velocity u at body of mass M, which is receding from it with velocity V ;

the bullet perforates the body and emerges with a velocity v. Show that the subsequent velocity of the body is

$$V + \frac{m(u-v)}{M}.$$

28. If two inelastic spheres have a direct impact, show that the K.E. lost by the impact is that of a body whose mass is half the harmonic mean between the masses of the two impinging spheres, and whose velocity is equal to their relative velocity before impact.

29. A shell of mass m is ejected from a gun of mass M by an explosion which generates kinetic energy E. Prove that the initial velocity of the shell is

$$\sqrt{\frac{2ME}{(M+m)m}}.$$

30. A smooth wedge of mass M and angle a is free to move on a smooth horizontal plane in a direction perpendicular to its edge. A particle of mass m is projected directly up the face of the wedge with velocity V. Prove that it returns to the point on the wedge from which it was projected after a time

$$2V\frac{(M+m\sin^2 a)}{(m+M)g\sin a}.$$

ANSWERS

1. 5 units ; 2 lbs. wt. **2.** 20 lbs. wt. **3.** 8 ft. per sec.

4. $7\frac{1}{2}$ ft. per sec. , 5775 ft.-lbs. **5.** $mv/(m+M)$.

6. 3600 ft. per sec. **7.** 204 lbs. wt. **8.** 9 inches.

9. 2 ft. ; $\frac{1}{2}$ sec. **10.** 1000 ft. per sec.

11. The masses move with a velocity of 24 ft./sec.

12. 5·2 ft./sec. **13.** $2\frac{1}{2}$ ft. ; 60 units of momentum.

14. $\frac{3}{4}$ sec. ; $\frac{11}{4}$ ft. **15.** $\frac{1}{12}$ lbs. wt.

16. 9·05 lbs. wt. nearly. **17.** 375 lbs. wt.

19. $\frac{(\sqrt{m_1x_1}+\sqrt{m_2x_2})^2}{m_1+m_2}$. **23.** $\frac{M}{M+m}\cdot\frac{u^2}{g}\sin 2a$.

CHAPTER XII X

COLLISION OF ELASTIC BODIES

12·1. A solid body has a definite shape. If a force is applied at any point of it trying to change its shape, in general all solids which we meet with in nature yield slightly and get more or less deformed near the point. Immediately, internal forces come into play tending to restore the body to its original form, and as soon as the disturbing force is removed, provided it is not too large, the body regains its original form. This property of a solid is referred to as its *elasticity of shape.*

If a ball be dropped from any height upon a hard floor, it is observed, after striking the floor, to rebound to a certain height (which is in general less than the height from which it is dropped). The reason for this is the elastic property of the solid referred to above. When the ball strikes the floor, it does not meet the floor at a single point. The impulsive action of the floor rapidly stops the downward velocity of the ball, and at the same time causes a temporary compression near the point of contact, and the ball actually meets the floor in a small circle, as can be verified by laying a thin layer of coloured powder on the floor. Now on account of the elastic property of the solid it tends to regain its original form quickly, and in so doing, presses the floor and receives an equal and opposite impulsive reaction from it, and thereby gains the upward velocity with which which it rebounds.

Now different substances have got their elastic properties different. If balls of different materials be dropped from the same height upon a floor, (or if the same ball be dropped from the same height upon floors of different constitution), the heights to which they rebound after striking the floor will be observed to be different.

Again, if the same ball be dropped on the same floor from different heights, the height of rebound will also vary, being greater when the ball is dropped from a greater height. Now, the height from which the ball is dropped gives us the velocity with which the ball meets the floor immediately before collision. Also the height to which the ball rebounds, gives us the velocity with which the ball started immediately after striking the floor. A remarkable thing may be noticed. It will be found that the ratio of the velocity with which the ball separates from the floor immediately after collision, to the velocity with which it approaches the floor immediately before striking it, is a constant so long as the ball and the floor are the same, whatever the height from which the ball is dropped, and this constant differs for different sets of ball and floor. Newton's experiments on the collision of two bodies (not simply of a ball on a floor, but between two balls both moving differently) lead to a similar result, which is formally enunciated in the next article.

12·2. Direct and oblique impact : Newton's law.

When two bodies come into collision, the common normal at their point of contact (to their touching surfaces) is known as the **line of impact.**

In case of two impinging spheres, clearly the line of centres is the line of impact.

When two impinging bodies have got their velocities immediately before collision both along the line of impact, it is said to be a case of **direct impact.**

When either of the colliding bodies has got its velocity immediately before collision in a direction different from the line of impact, the case is one of **oblique impact.**

Newton's Experimental law on Collision.

When two bodies impinge on one another, the relative velocity of separation of the two bodies immediately after

impact, measured along the line of impact, bears a constant ratio to their relative velocity of approach along the same direction immediately before impact.

The constant ratio (usually denoted by e) for a particular pair of colliding bodies is referred to as their **coefficient (or modulus) of elasticity** (or **restitution**, or **resilience**).

Mathematically speaking, if u_1, u_2 be the camponents of velocity of two colliding bodies along their line of impact before collision, and v_1, v_2 their component velocities after collision along the same line, all measured in the *same sense*, and e be the coefficient of restitution, then

$$\frac{v_2 - v_1}{u_1 - u_2} = e \text{ or } v_2 - v_1 = -e\,(u_2 - u_1).$$

Thus, the greater the relative velocity with which two bodies strike each other, the greater is the relative velocity with which they separate.

When one or both the bodies are altered, e becomes different, but so long as both the bodies remain the same, e is constant.

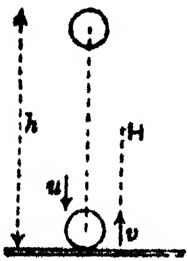
The quantity e, which is a positive number, is never greater than unity. When for a pair of colliding bodies $e = 1$, that is when the relative velocity of separation of two bodies after collision is equal to their relative velocity of approach immediately before the impact, the bodies are said to be **perfectly elastic.**

When for a pair of colliding bodies $e = 0$, that is when two bodies after collision do not separate, they are said to be **inelastic.**

Perfectly elastic bodies are never met with in nature. A very good approach is a pair of glass balls for which $e = \cdot 94$.

12·3. Direct impact of a sphere on a fixed plane.

A ball is dropped from a height h on a horizontal floor. The coefficient of elasticity between the ball and the floor being e, to find the height to which the ball rebounds.



The velocity u acquired by the ball in falling freely under gravity through a height h is given by

$$u^2 = 2gh \text{ or } u = \sqrt{2gh}.$$

This is then the velocity of approach immediately before collision with which the ball strikes the floor.

Let v be the upward velocity with which the ball separates from the floor immediately after collision. Both u and v, being vertical, are along the common normal at the point of contact of the ball with the floor, and so the impact is direct.

By Newton's experimental law of impact,

$$v = eu = e\sqrt{2gh}.$$

Hence, H being the height to which the ball rebounds,

$$0 = v^2 - 2gH,$$

$$\text{or,} \quad H = \frac{v^2}{2g} = \frac{e^2.2gh}{2g} = e^2 h.$$

Note 1. The above discussion indicates a rough method of determining e for bodies of any two materials. A thick plate of one material is fixed on the floor, while a ball is made of the other; h and H are observed against the graduations on a neighbouring vertical wall, when $e = \sqrt{H/h}$.

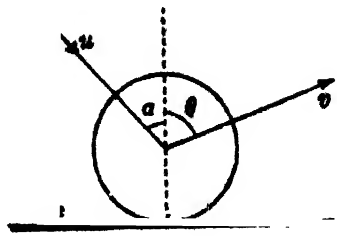
Note 2. *Impulse of the blow* = change of the momentum of the ball upwards $= m\{eu - (-u)\} = mu(1+e)$ where m is the mass of the ball.

12·4. Oblique impact of a smooth sphere on a fixed plane.

A smooth sphere impinges obliquely on a fixed plane with a velocity u at an angle α with the line of impact, e being the

coefficient of restitution between the sphere and the plane; to find the velocity immediately after impact.

Let v be the velocity of the sphere immediately after the impact, in a direction making an angle θ with the line of impact.



The component of velocity along the line of impact immediately before the impact is $u \cos \alpha$ towards the floor, which is thus the velocity of approach along the line of impact. Similarly, the velocity of separation from the floor measured along the same line after impact is given by $v \cos \theta$.

Hence, by Newton's experimental law of impact,

$$v \cos \theta = eu \cos \alpha. \qquad \cdots \quad \text{(i)}$$

Again, since the sphere is smooth, the impulsive reaction of the floor on the ball is along the common normal, that is, along the line of impact, and the change of velocity of the ball will be produced in this direction only. Perpendicular to the line of impact, (that is parallel to the floor) there being no force component, the component velocity in that direction will remain unchanged, and so

$$v \sin \theta = u \sin \alpha. \qquad \cdots \quad \text{(ii)}$$

From (i) and (ii), we get

$$v = u\sqrt{\sin^2\alpha + e^2 \cos^2\alpha}, \quad \theta = \tan^{-1}\left(\frac{\tan \alpha}{e}\right).$$

Cor. If $e = 1$, then $v = u$ and $\theta = \alpha$.

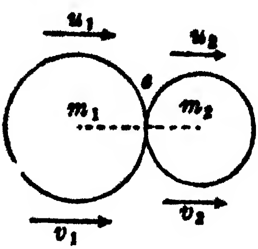
Hence, a perfectly elastic ball impinging obliquely on a fixed plane rebounds with the same velocity, making the same angle with the normal to the plane as the angle of incidence.

Note. Here, *impulse of the blow* = change of momentum of the ball along the line of impact $= m\{v \cos \theta - (-u \cos \alpha)\} = m(1+e)\, u \cos \alpha$ from (i), m being the mass of the ball.

Again, from (i) and (ii), $v^2 = u^2(\sin^2\alpha + e^2 \cos^2\alpha)$. Hence, *the loss of K.E. by the impact* $= \frac{1}{2}mu^2 - \frac{1}{2}mv^2 = \frac{1}{2}(1-e^2)\, mu^2 \cos^2\alpha$.

12·5. Direct impact of two smooth spheres.

Two smooth spheres of masses m_1 and m_2 moving along their line of centres with velocities u_1 and u_2 (measured in the same sense) impinge directly. To find their velocities immediately after impact, e being the coefficient of restitution between them.



Let v_1 and v_2 be the velocities of the two spheres immediately after impact measured along their line of centres in the same direction in which u_1 and u_2 are measured. As the spheres are smooth, the impulsive action and reaction between them will be along the common normal at the point of contact *i.e.*, along their line of centres, and so perpendicular to this line no velocity will be generated in the spheres.

From the principle of conservation of momentum,

$$m_1v_1 + m_2v_2 = m_1u_1 + m_2u_2. \quad \cdots \quad \text{(i)}$$

Also from Newton's experimental law of impact of two bodies, e denoting the coefficient of restitution between the spheres,

$$v_2 - v_1 = e(u_1 - u_2). \quad \cdots \quad \cdots \quad \text{(ii)}$$

Multiplying (ii) by m_2 and subtracting from (i),

$$(m_1 + m_2)v_1 = (m_1 - em_2)u_1 + m_2u_2(1+e). \quad \cdots \quad \text{(iii)}$$

Similarly, multiplying (ii) by m_1 and adding to (i),

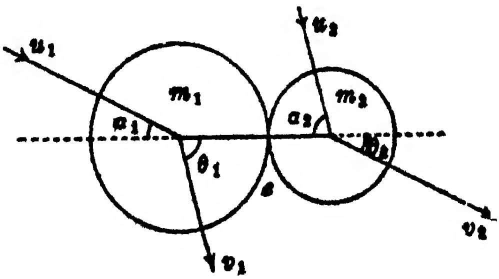
$$(m_1 + m_2)v_2 = m_1u_1(1+e) + (m_2 - em_1)u_2. \quad \cdots \quad \text{(iv)}$$

(iii) and (iv) give v_1 and v_2 respectively.

Cor. If $m_1 = m_2$ and $e = 1$, we get $v_1 = u_2$ and $v_2 = u_1$. Hence, *two equal perfectly elastic spheres after direct impact interchange their velocities.*

12·6. Oblique impact of two smooth spheres.

Let two smooth spheres of masses m_1 and m_2, moving with velocities u_1 and u_2 at angles α_1 and α_2 with their line of centres, come into collision, as indicated in the figure.



Let v_1 and v_2 be their velocities immediately after impact in directions making angles θ_1 and θ_2 respectively with their line of centres.

As the bodies are smooth, the impulsive action and reaction between them will be along their line of centres, and so perpendicular to this direction there will be no change in the component velocities. Hence,

$$v_1 \sin \theta_1 = u_1 \sin \alpha_1 \qquad \cdots \qquad \cdots \quad \text{(i)}$$

$$v_2 \sin \theta_2 = u_2 \sin \alpha_2. \qquad \cdots \quad \text{(ii)}$$

Again, by the principle of conservation of momentum, the sum-total of the momenta of the two bodies along the line of impact will be unchanged. Hence,

$$m_1 v_1 \cos \theta_1 + m_2 v_2 \cos \theta_2 = m_1 u_1 \cos \alpha_1 + m_2 u_2 \cos \alpha_2. \qquad \cdots \quad \text{(iii)}$$

Lastly, by Newton's experimental law of impact,

$$\underbrace{v_2 \cos \theta_2 - v_1 \cos \theta_1}_{\text{(the relat. velocity of separation along the line of impact)}} = e\underbrace{(u_1 \cos \alpha_1 - u_2 \cos \alpha_2)}_{\text{(the relat. velocity of approach along the line of impact)}} \quad \cdots \quad \text{(iv)}$$

From (iii) and (iv), we get

$$v_1 \cos\theta_1 = \frac{(m_1 - em_2)u_1 \cos\alpha_1 + (1+e)m_2 u_2 \cos\alpha_2}{m_1 + m_2} \quad \dots \text{(v)}$$

$$v_2 \cos\theta_2 = \frac{(1+e)m_1 u_1 \cos\alpha_1 + (m_2 - em_1)u_2 \cos\alpha_2}{m_1 + m_2} \quad \dots \text{(vi)}$$

From (i) and (v), we get v_1 and θ_1, and similarly from (ii) and (vi), v_2 and θ_2 are obtained.

12·7. Loss of Energy due to collision.

(A) *Direct impact.*

Let two smooth spheres of masses m_1 and m_2 moving with velocities u_1 and u_2 along their line of centres come into direct collision, and let v_1 and v_2 be their velocities immediately after impact measured along the same direction. The spheres being smooth, the impulsive action and reaction between them will be along the line of centres, and so perpendicular to that line there will be no velocity generated in the spheres. Let e be the coefficient of restitution between the spheres.

Then by the principle of conservation of momentum,

$$m_1 v_1 + m_2 v_2 = m_1 u_1 + m_2 u_2. \quad \dots \text{(i)}$$

Also, by Newton's experimental law of impact,

$$v_2 - v_1 = e(u_1 - u_2). \quad \dots \text{(ii)}$$

Now, the loss of the total kinetic energy of the spheres

$$= (\tfrac{1}{2}m_1 u_1^2 + \tfrac{1}{2}m_2 u_2^2) - (\tfrac{1}{2}m_1 v_1^2 + \tfrac{1}{2}m_2 v_2^2)$$

$$= \frac{1}{2(m_1+m_2)}\{(m_1+m_2)(m_1 u_1^2 + m_2 u_2^2)$$

$$-(m_1+m_2)(m_1 v_1^2 + m_2 v_2^2)\}. \quad \dots \text{(iii)}$$

But $(m_1+m_2)(m_1 u_1^2 + m_2 u_2^2)$

$$= m_1^2 u_1^2 + m_2^2 u_2^2 + m_1 m_2 (u_1^2 + u_2^2)$$

$$= (m_1 u_1 + m_2 u_2)^2 + m_1 m_2 (u_1 - u_2)^2.$$

Similarly, $(m_1+m_2)(m_1v_1^2+m_2v_2^2)$

$$=(m_1v_1+m_2v_2)^2+m_1m_2\,(v_1-v_2)^2$$
$$=(m_1u_1+m_2u_2)^2+m_1m_2.e^2(u_1-u_2)^2.$$

[*from (i) and (ii)*]

Hence, substituting in (iii),

the loss of total kinetic energy

$$=\frac{1}{2\,(m_1+m_2)}\{(m_1u_1+m_2u_2)^2+m_1m_2\,(u_1-u_2)^2$$
$$-(m_1u_1+m_2u_2)^2-m_1m_2.e^2\,(u_1-u_2)^2\}$$
$$=\frac{1}{2}\,\frac{m_1m_2}{m_1+m_2}\,(1-e^2)(u_1-u_2)^2.$$

As $e<1$ generally, the above expression is essentially positive, and thus there is actually a loss of total K.E. by a collision. Only in the case $e=1$, *i.e.*, in case of a collision of perfectly elastic bodies, the above expression is zero, and hence the total K.E. is unchanged by impact.

(B) *Oblique impact.*

Let two smooth spheres impinge obliquely one on another, and let the notation be as in Art. 12·6.

The loss of the total K.E. of the spheres by the impact

$$=\tfrac{1}{2}\,(m_1u_1^2+m_2u_2^2)-\tfrac{1}{2}\,(m_1v_1^2+m_2v_2^2)$$
$$=\tfrac{1}{2}\{m_1u_1^2\,(\cos^2a_1+\sin^2a_1)+m_2u_2^2\,(\cos^2a_2+\sin^2a_2)\}$$
$$-\tfrac{1}{2}\{m_1v_1^2\,(\cos^2\theta_1+\sin^2\theta_1)+m_2v_2^2\,(\cos^2\theta_2+\sin^2\theta_2)\}$$
$$=\tfrac{1}{2}\{m_1u_1^2\,\cos^2a_1+m_2u_2^2\,\cos^2a_2\}$$
$$-\tfrac{1}{2}\,\{m_1v_1^2\,\cos^2\theta_1+m_2v_2^2\,\cos^2\theta_2)\}$$

[*by equations (i) and (ii) of Art. 12·6*]

Now, with the help of equations (iii) and (iv) of Art. 12·6, proceeding exactly as in the above case (A) of direct impact (noting that u_1, u_2, v_1, v_2 of the case of direct impact are only replaced by $u_1\cos a_1$, $u_2\cos a_2$, $v_1\cos\theta_1$, $v_2\cos\theta_2$ in this case) the expression for the loss of total K.E. by impact becomes

$$\frac{1}{2}\,\frac{m_1m_2}{m_1+m_2}\,(1-e^2)(u_1\cos a_1-u_2\cos a_2)^2,$$

which is essentially positive, since $e < 1$. Here also, for $e = 1$, *i.e.*, for oblique impact of two perfectly elastic smooth spheres, there is no alteration in the total kinetic energy.

12·8. Impulsive action (or reaction) **between two colliding spheres.**

(A) *When there is a direct impact.*

Let m_1 and m_2 be the masses of two smooth spheres coming into a direct collision, u_1 and u_2 being their respective velocities immediately before, and v_1 and v_2 their velocities immediately after the impact along the line of centres, all measured in the same sense.

Then, from the principle of conservation of momentum,

$$m_1v_1 + m_2v_2 = m_1u_1 + m_2u_2. \qquad \cdots \quad \text{(i)}$$

Also, from Newton's experimental law of impact,

$$v_2 - v_1 = e(u_1 - u_2). \qquad \cdots \quad \text{(ii)}$$

Multiplying (ii) by m_1 and adding to (i),

$$(m_1 + m_2)v_2 = m_1u_1 + m_2u_2 + em_1(u_1 - u_2). \quad \cdots \quad \text{(iii)}$$

Now the impulsive blow on m_2, measured by its impulse, being equal to the change of momentum produced, is

$$I = m_2(v_2 - u_2)$$

$$= \frac{m_2}{m_1 + m_2}\left[m_1u_1 + m_2u_2 + em_1(u_1 - u_2) - (m_1 + m_2)u_2\right]$$

[*using (iii)*]

$$= \frac{m_2}{m_1 + m_2}\left[m_1(u_1 - u_2) + em_1(u_1 - u_2)\right]$$

$$= \frac{m_1m_2}{m_1 + m_2}(1 + e)(u_1 - u_2).$$

The impulsive blow on m_1 is equal and opposite to it.

(B) *When there is an oblique impact.*

The impulse of the blow, being measured by the change of momentum produced along the line of impact, we are

to take into consideration only the components of the initial and final velocities of the spheres along this line, and proceeding exactly as in the above case, we get the measure of the required impulsive blow in this case given by

$$I = m_2 (v_2 \cos \theta_2 - u_2 \cos \alpha_2)$$
$$= \frac{m_1 m_2}{m_1 + m_2} (1+e)(u_1 \cos \alpha_1 - u_2 \cos \alpha_2).$$

12·9. Illustrative Examples.

Ex. 1. *A ball overtakes another ball of m times its mass, which is moving with $\frac{1}{n}$th of its velocity in the same direction. If the impact reduces the first ball to rest, prove that the coefficient of restitution is*

$$\frac{m+n}{mn-m},$$

and that m must not be less than $\frac{n}{n-2}$.

Let M be the mass and V the velocity before impact of the first ball; then mM and $\frac{V}{n}$ are the corresponding quantities for the second. Let v be the velocity after impact of the second ball, the first ball being reduced to rest.

Now e being the coefficient of restitution, we get, from Newton's experimental law,

$$v = e\left(V - \frac{V}{n}\right) = eV \cdot \frac{n-1}{n}. \quad \ldots \qquad \text{(i)}$$

Also, from the principle of conservation of momentum,

$$mMv = MV + mM \cdot \frac{V}{n},$$

or, $$v = \frac{V}{m} + \frac{V}{n} = V \cdot \frac{m+n}{mn}.$$

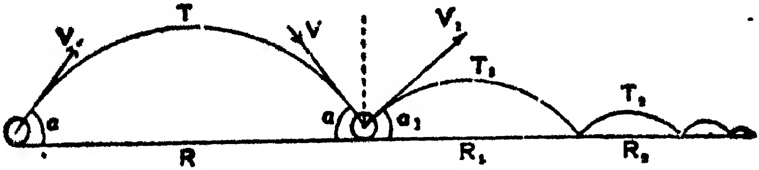
Hence, from (i),

$$eV \cdot \frac{n-1}{n} = V \cdot \frac{m+n}{mn}, \quad \text{or,} \quad e = \frac{m+n}{m(n-1)} = \frac{m+n}{mn-m}.$$

Again, as e is never greater than unity, $mn - m \nless m+n$, *i.e.*, $m(n-2) \nless n$; in other words, m is not less than $\frac{n}{n-2}$.

Ex. 2. *A ball is thrown from a point on a smooth horizontal ground with a velocity V at an angle α to the horizon. Assuming e to be the coefficient of restitution of the ball with the ground, show that the total time for which the ball rebounds on the ground is $\frac{2V \sin \alpha}{g(1-e)}$, and that its distance from the starting point when it ceases to rebound is $\frac{V^2 \sin 2\alpha}{g(1-e)}$.*

[*C. U. 1948*]



We know that the projected ball, after describing a parabolic orbit will strike the ground with the same velocity V, at the same angle α with the horizon, as at start. The time of flight $T = \frac{2V \sin \alpha}{g}$, and the range $R = \frac{2V^2}{g} \sin \alpha \cos \alpha$.

Now, suppose that immediately after the first rebound, the velocity of the ball is V_1 at angle α_1 to the horizon. Then by Newton's law of impact, considering motion along the line of impact, *i.e.*, in the vertical direction,

$$V_1 \sin \alpha_1 = e.V \sin \alpha$$

and as perpendicular to the line of impact there is no change of velocity,

$$V_1 \cos \alpha_1 = V \cos \alpha.$$

For the parabolic orbit after the first rebound till the second rebound,

$$\text{the time of flight } T_1 = \frac{2V_1 \sin \alpha_1}{g} = \frac{2eV \sin \alpha}{g} = eT$$

$$\text{and the range } R_1 = \frac{2V_1^2 \sin \alpha_1 \cos \alpha_1}{g} = \frac{2eV^2 \sin \alpha \cos \alpha}{g} = eR.$$

Similarly, from the second to the third rebound,

$$T_2 = eT_1 = e^2 T, \text{ and } R_2 = eR_1 = e^2 R;$$

and so on.

Hence, total time for which the ball rebounds

$$=T+eT+e^2T+\cdots \text{ ad inf}=\frac{T}{1-e}=\frac{2V\sin\alpha}{g(1-e)}$$

and the total distance moved over by the ball before it ceases to rebound

$$=R+eR+e^2R+\cdots \text{ad inf}=\frac{R}{1-e}=\frac{V^2\sin 2\alpha}{g(1-e)}.$$

Ex. 3. *A ball impinges directly upon another ball at rest, and is itself reduced to rest by the impact; if half of the initial kinetic energy is destroyed in the collision, find the coefficient of restitution.*

[*U. P. 1941*]

Let m be the mass of the first ball, u its velocity immediately before impact, M the mass of the second ball, and V its velocity immediately after impact. Then from the principle of conservation of momentum and by Newton's law of impact, we get respectively,

$$MV=mu \text{ and } V=eu. \quad \cdots \quad \text{(i)}$$

Now, by the given condition, energy lost = half the initial energy,

$$\text{or,} \quad \tfrac{1}{2}mu^2-\tfrac{1}{2}MV^2=\tfrac{1}{2}(\tfrac{1}{2}mu^2).$$

$$\text{or,} \quad 1-\frac{MV^2}{mu^2}=\tfrac{1}{2},$$

i.e., using (i),

$$1-e=\tfrac{1}{2}, \quad \text{or,} \quad e=\tfrac{1}{2}.$$

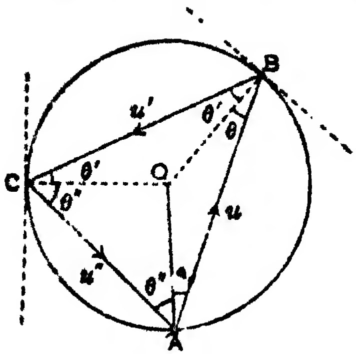
Ex. 4. *A smooth circular hoop lies on a smooth horizontal table, and is held fixed. A particle is projected on the table from a point on the inner circumference of the hoop. Prove that if the particle returns to the position of projection on the hoop after two impacts, its original direction of projection must make with the radius through the point, an angle,*

$$\tan^{-1}\{e^3/(1+e+e^2)\}^{\frac{1}{2}},$$

e being the coefficient of restitution.

Prove that the same result is true even if the hoop is free to move on the table.

Let u be the velocity with which the particle is projected from A in a direction AB at an angle θ to the radius OA.



After its first impact at B let u' be its velocity along BC at an angle θ' with OB. After its second impact at C, the particle moves along CA to return to its starting position A. Let θ'' be the angle OCA.

Then for the first impact at B, since OB is the line of impact, by Newton's law,

$$u' \cos \theta' = e.u \cos \theta \quad \cdots \text{(i)}$$

and perpendicular to the line of impact, there being no change of velocity,

$$u' \sin \theta' = u \sin \theta. \quad \cdots \text{(ii)}$$

From (i) and (ii), $\tan \theta' = \dfrac{\tan \theta}{e}$.

Similarly, for the second impact,

$$\tan \theta'' = \frac{\tan \theta'}{e} = \frac{\tan \theta}{e^2}.$$

Now from Geometry, since $OA = OB = OC$, from triangle ABC,

$$2\theta + 2\theta' + 2\theta'' = \pi, \quad \text{or,} \quad \theta' + \theta'' = \frac{\pi}{2} - \theta.$$

$$\therefore \quad \tan(\theta' + \theta'') = \cot \theta, \text{ or, } \frac{\tan \theta' + \tan \theta''}{1 - \tan \theta' \tan \theta''} = \cot \theta,$$

i.e., $\tan \theta (\tan \theta' + \tan \theta'') = 1 - \tan \theta' \tan \theta''$,

or, $\tan \theta \left(\dfrac{\tan \theta}{e} + \dfrac{\tan \theta}{e^2} \right) = 1 - \dfrac{\tan^2 \theta}{e^3}$,

or, $\tan^2 \theta \left(\dfrac{1}{e} + \dfrac{1}{e^2} + \dfrac{1}{e^3} \right) = 1$ giving $\tan^2 \theta = \dfrac{e^3}{1 + e + e^2}$,

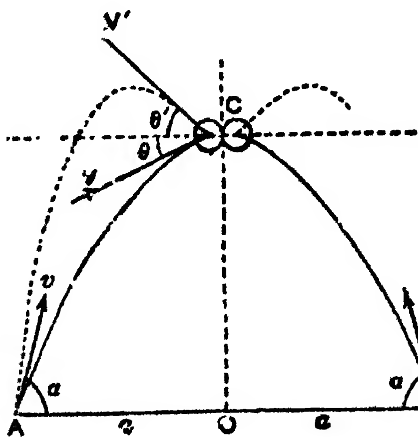
whence, $\theta = \tan^{-1} \{e^3/(1 + e + e^2)\}^{\frac{1}{2}}$.

If the hoop is free to move on the table, we assume u' to be the velocity of the particle relative to the hoop after impact at B, and BC at an angle θ' to OB, the relative direction of motion. The equations (i) and (ii) are then unaltered. Again, after the second impact with the hoop at the point C of the hoop, the relative direction of motion of the particle with respect to the hoop is CA, in order that it may come to the same position A on the hoop (though not the same position in space) as at start. As Newton's law relates to relative velocities of approach and separation between the two bodies, the equations of impact are unaltered. Thus, the final result is the same.

Ex. 5. *Two equal elastic balls are projected towards each other at the same instant in the same vertical plane from two points in the same horizontal plane, v beieg the velocity and α the elevation in each case. Show that after impact they will return to the points of projection if*

$$ga(1+e)=ev^2 \sin 2\alpha,$$

e being the coefficient of restitution, and $2a$ the distance between the points of projection,



Let the balls projected from A and B come to collision at C, which by symmetry is vertically above the mid-point O of AB, at a height h say. Let V be the velocity of either at an angle θ to the horizon

immediately before impact, and V' at an angle θ' to the horizon, that immediately after. Let t be the time from A to C, and t' the time in the subsequent parabolic path from C to A of one of the balls. The line of impact is clearly horizontal, and considering the motion along and perpendicular to the line of impact,

$$2V' \cos\theta' = e.2V \cos\theta \text{ and } V' \sin\theta' = V \sin\theta. \quad \cdots \quad \text{(i)}$$

Now, for the initial parabolic path from A to C,

$$\left.\begin{aligned} V\cos\theta &= v\cos\alpha \\ V\sin\theta &= v\sin\alpha - gt \end{aligned}\right\} \quad \cdots \quad \cdots \quad \text{(ii)}$$

and
$$a = v\cos\alpha . t \quad \cdots \quad \cdots \quad \text{(iii)}$$

$$h = v\sin\alpha . t - \tfrac{1}{2}gt^2. \quad \cdots \quad \cdots \quad \text{(iv)}$$

Also for the subsequent parabolic path from C to A,

$$a = V'\cos\theta' . t' = e . V\cos\theta . t' = e . v\cos\alpha . t' \quad \cdots \quad \text{(v)}$$

$$-h = V'\sin\theta' . t' - \tfrac{1}{2}gt'^2 = V\sin\theta . t' - \tfrac{1}{2}gt'^2$$

$$= (v\sin\alpha - gt)\, t' - \tfrac{1}{2}gt'^2. \quad \cdots \quad \cdots \quad \text{(vi)}$$

From (iii) and (v),

$$(t + t') = \frac{a}{v\cos\alpha}\left(1 + \frac{1}{e}\right). \quad \cdots \quad \cdots \quad \text{(vii)}$$

Also from (iv) and (vi), adding,

$$0 = v\sin\alpha\,(t + t') - \tfrac{1}{2}g\,(t + t')^2 \quad \cdots \quad \cdots \quad \text{(viii)}$$

whence $2v\sin\alpha = g\,(t+t') = \dfrac{ga}{v\cos\alpha} \cdot \dfrac{1+e}{e}$ [*by (vii)*]

$\therefore\ ga(1+e) = 2ev^2 \sin\alpha\cos\alpha = ev^2 \sin 2\alpha.$

Note. It may be noted that by the impact at C, the vertical component of velocity is not altered. Hence, considering the continuous motion from A to C and back from C to A, so far as the vertical motion is concerned, we see that in time $t + t'$, the vertical distance described by the ball is zero. Hence, we can at once write down the equation (viii), instead of writing the equations (iv) and (vi) and thence deducing (viii) with the help of (iii) and (v).

Ex. 6. *Prove that the kinetic energy of two particles of masses m and m′ moving in a plane is*

$$\tfrac{1}{2}(m+m')V^2+\tfrac{1}{2}\frac{mm'v^2}{m+m'},$$

where V is the velocity of the centre of mass of the particles and v the velocity of either of them relative to the other. [*C. H. 1966, 1968*]

With reference to a set of rectangular axes OX, OY in the plane, let u_1, v_1 and u_2, v_2 be the components of the velocities of the masses m and m' parallel to OX, OY and let U_1, V_1 be the corresponding components of the velocity of their C.M. Then we have

$$U_1=\frac{mu_1+m'u_2}{m+m'} \quad \cdots (1) \quad V_1=\frac{mv_1+m'v_2}{m+m'}. \quad \cdots (2)$$

And, $U_1^2+V_1^2=V^2$. $\cdots$ (3)

Also $(u_1-u_2)^2+(v_1-v_2)^2=v^2$. $\cdots$ (4)

From (1) and (2),

$$(mu_1+m'u_2)^2+(mv_1+m'v_2)^2=(m+m')^2(U_1^2+V_1^2)$$
$$=(m+m')^2V^2 \text{ by (3)} \quad \cdots (5)$$

Multiplying (4) by mm', we get

$$mm'u_1^2+mm'u_2^2-2mm'u_1u_2+mm'v_1^2+mm'v_2^2-2mm'v_1v_2$$
$$=mm'v^2. \quad \cdots (6)$$

Adding (5) and (6), we get

$$(m+m')\{m(u_1^2+v_1^2)+m'(u_2^2+v_2^2)\}$$
$$=mm'v^2+(m+m')^2V^2.$$

$$\therefore \ \tfrac{1}{2}m(u_1^2+v_1^2)+\tfrac{1}{2}m'(u_2^2+v_2^2)=\tfrac{1}{2}\frac{mm'}{m+m'}v^2+\tfrac{1}{2}(m+m')V^2.$$

$$\therefore \ \text{K.E. of } m+\text{K.E. of } m'=\tfrac{1}{2}\frac{mm'}{m+m'}v^2+\tfrac{1}{2}(m+m')V^2.$$

Examples on Chapter XII

1. A particle falls from a certain height upon a fixed horizontal plane and rebounds, and takes 1 second to reach the plane again. If the coefficient of restitution be $\frac{1}{2}$, find the height from which it fell.

2. A glass marble projected along the smooth floor of a room hits directly the opposite wall and returns to the point of projection again. If it takes thrice as long in returning as it took in going, find the coefficient of elasticity.

3. A ball projected vertically upwards with a velocity of 32 ft. per second from the ground meets with an obstacle at a height of 4 ft. and returns to the ground again. If the coefficient of restitution between the ball and the obstacle be $\frac{1}{\sqrt{3}}$, and that between the ball and the ground be $\frac{1}{\sqrt{2}}$, show that the ball after rebounding from the ground will again just reach the height of the obstacle.

4. A ball is dropped on a horizontal floor. If the time taken by the ball to rise to the greatest height from the floor after the second impact be half of that taken by the ball to drop down to the floor before the first impact, show that $e = \sqrt{\frac{1}{2}}$.

5. A ball falls from a height of 36 feet upon an elastic horizontal plane. If the coefficient of elasticity be $\frac{1}{3}$, find the total space described before the ball ceases rebounding.

✓ 6. A particle falls from a height h upon a fixed horizontal plane. If e be the coefficient of restitution, show that the whole distance described before the particle has finished rebounding is

$$\frac{1+e^2}{1-e^2}h,$$

and that the whole time taken is

$$\sqrt{\frac{2h}{g}} \cdot \frac{1+e}{1-e}.$$

7. A ball of mass 5 ozs. moving with a velocity of 48 ft. per second, impinges on a fixed smooth plane in a direction making an angle of 60° with the plane. If the coefficient

of restitution be $\frac{1}{3}$, find the velocity and direction of motion of the ball after the impact.

Find also the impulsive action on the plane.

8. A ball falls from a height of 25 feet upon an inclined plane of elevation 45°. If the coefficient of restitution be $\frac{1}{3}$, find the magnitude and direction of the velocity after impact.

9. A ball slides from rest from the top of a smooth inclined plane of height h, and elevation 45°. If the coefficient of restitution be $\frac{1}{2}$, show that the range on the horizontal plane after rebound is h.

10. A perfectly elastic ball dropped on an inclined plane strikes the plane again after rebounding. Show that the interval between the times of the two impacts is independent of the inclination of the plane.

11. A particle is dropped from a height of 16 feet on a plane of elevation of 30°. How far down the plane is its next point of impact, if the coefficient of restitution be $\frac{1}{4}$?

12. A sphere of mass 5 lbs. moving with a velocity of 6 ft. per sec. overtakes a sphere of mass 4 lbs. moving with a velocity of 4 ft. per sec. in the same direction ; if the impact be direct and the coefficient of restitution be $\frac{1}{3}$, find the velocities of the spheres after impact. Find also the impulse of the blow.

13. A ball moving with a velocity of 8 ft. per second impinges directly on an equal ball moving in the same line with a velocity of 4 ft. per sec. in the opposite direction ; if the coefficient of restitution be $\frac{1}{3}$, show that after impact the first is reduced to rest and the second turns back in the opposite direction with the velocity it had before impact.

14. Two equal spheres whose elasticity is $\frac{3}{4}$ moving in opposite directions with velocities 8 cms. and 4 cms. per sec. respectively, impinge directly upon each other. Find the distance apart between the spheres 10 secs. after the impact.

15. Two balls impinge directly and the impact interchanges their velocities ; prove that they are perfectly elastic and of equal masses.

16. A ball A directly strikes a ball B which is at rest and after collision their velocities are equal and opposite ; find the coefficient of elasticity, supposing the mass of B to be k times the mass of A.

Is there any restriction on the value of k ?

17. A sphere impinges directly on an equal sphere at rest ; if the coefficient of restitution be e, show that their velocities after the impact are as $1-e : 1+e$.

18. An elastic pile weighing w lbs., is driven vertically by a hammer weighing W lbs., the hammer having a fall of x feet. If the resistance to penetration be P lbs. wt. and e the coefficient of elasticity, find the distance penetrated by the pile into the ground.

19. A ball is dropped from a height of 48 ft., and at the same instance an equal ball is projected vertically upwards from the ground with a velocity of 96 ft. per sec. The two balls collide with one another. If the coefficient of restitution be $\frac{1}{2}$, find the times taken by the balls to reach the ground after the impact.

20. An elastic ball of mass m falls from a height h on a fixed horizontal plane and rebounds. Show that the loss of K.E. by the impact is $mgh(1-e^2)$. [*C. U. 1930, '49*]

21. Two balls of masses 2 lbs. and 3 lbs. are moving with velocities 6 ft. per sec. and 3 ft. per sec. respectively in the same direction along the same straight line and collide with one another. Find the K.E. lost by impact if the coefficient of restitution be $\frac{2}{3}$.

22. Two equal balls are moving in the same direction along the same straight line with velocities, one double of the other. They collide and lose by impact $\frac{1}{16}$th of their kinetic energy. Find the coefficient of restitution.

23. If two unequal spherical balls moving with equal velocities v in opposite directions impinge directly, prove that the resulting loss of K.E. is $(1-e^2)\mu v^2$, where μ is the harmonic mean between the masses of the balls.

24. Two perfectly inelastic bodies of masses m_1 and m_2 moving with velocities u_1 and u_2 in the same direction impinge directly. Show that the loss of K.E. due to impact is

$$\frac{1}{2}\frac{m_1 m_2}{m_1+m_2}(u_1-u_2)^2. \qquad [\,C.\ U.\ 1939\,]$$

25. A sphere of mass 16 lbs. moving with velocity 8 ft. per sec. impinges on a fixed plane, in a direction making an angle of 30° with the plane. If the coefficient of restitution be $\frac{1}{4}$, find the loss of kinetic energy.

26. Three perfectly elastic balls A, B, C of masses 1, 2, 3 lbs. are moving in the same straight line with velocities 8, 2, 1 ft. per sec. respectively. A impinges upon B, and then B upon C. Show that after impact A and B are reduced to rest and C moves on with a velocity of 5 ft. per second.

27. There are n perfectly elastic equal balls at rest in a straight line; the first impinges directly on the second with velocity v, the second on the third, and so on. Show that the nth ball moves off with velocity v. What happens if the balls are imperfectly elastic, the coefficient of elasticity being e?

28. A ball of mass 4 lbs. moving with a velocity of 10 ft. per sec. strikes a ball of equal mass lying at rest. The impinging ball moves at an angle of 30° with the line of centres at the instant of impact, and the coefficient of elasticity is $\frac{1}{3}$. Find the velocity of the second ball after impact.

Find also the loss of K.E. by impact.

29. A ball of mass 2 grammes, moving with a velocity of 8 cms. per second impinges on a ball of mass 4 grammes moving with a velocity of 2 cms. per sec. If their velocities before impact be in like parallel directions and inclined at an angle of 30° to the line of centres at the instant of impact, find their velocities after impact, the coefficient of restitution being $\frac{1}{3}$.

Find also the impulsive action between the two bodies.

30. Two perfectly elastic equal balls impinge; show that if their direction of motion before impact be at right angles to each other, then their directions of motion after impact are also at right angles to each other.

31. (i) A sphere of mass m_1 impinges obliquely on a sphere of mass m_2 which is at rest. If $m_1 = em_2$, show that their directions of the motion after impact are at right angles.

(ii) A smooth sphere of mass m impinges on another of mass M at rest, the direction of motion making an angle of $45°$ with the line of centres at the moment of impact. If the coefficient of restitution be $\frac{1}{2}$, show that the direction of motion of m is turned through an angle

$$\tan^{-1}\frac{3M}{M+4m}. \qquad [\text{ C. H. 1962, '65 }]$$

32. If two equal and perfectly elastic spheres impinge obliquely, show that they interchange their velocities in the direction of their line of centres.

33. Two equal balls of elasticity $\frac{1}{2}$ start at the same instant with equal velocities from the opposite corners of a square along two contiguous sides and collide. Show that after collision their directions of motion are inclined at an angle $\tan^{-1}\frac{4}{3}$.

34. A ball whose coefficient of elasticity is e, is projected with a velocity u at an angle a to the horizon from a point in a horizontal plane. It strikes a fixed vertical wall situated at a distance of h feet and returns to the point of projection. Show that

$$1+\frac{1}{e}=\frac{u^2\sin 2a}{gh}.$$

35. There are two parallel walls, the distance between which is equal to their height; from the top of one of them a perfectly elastic ball is thrown horizontally, so as to fall at the foot of the same wall, after rebounding from the other. Show that the focus of the first path is at the foot of the first wall.

36. Hailstones are observed to strike the surface of a frozen lake in a direction making an angle of 30° with the vertical and to rebound at an angle of 60°. Assuming the contact to be smooth, find the coefficient of elasticity.

If the hailstones rise after impact to a height of 2 feet, find the velocity with which they originally struck the ground.

37. Two equal balls P and Q lie in contact in a horizontal circular groove. They are projected along the groove and come into collision after a time t. Show that the second impact takes place after a further interval of t/e, where e is the coefficient of elasticity.

38. A ball whose mass is 4 ounces impinges directly on a fixed plane with a velocity of 30 ft. per sec. If the coefficients of restitution be $\frac{2}{3}$ and the time of contact is $\frac{1}{256}$th of a second, find the average pressure between the ball and the plane in lbs. wt.

39. Two spheres of masses m_1 and m_2 travelling with velocities u_1 and u_2 in the same direction, collide directly, and rebound. If the velocities after impact are v_1 and v_2, and if e be the coefficient of restitution, show that each sphere loses the same amount of energy if

$$u_1 + u_2 + v_1 + v_2 = 0.$$

40. A series of perfectly elastic balls are arranged in the same straight line ; one of them impinges directly on the next and so on. Prove that if their masses form a geometrical progression of which the common ratio is r, their velocities after impact will form a geometrical progression of which the common ratio is $\frac{2}{1+r}$.

41. A perfectly elastic ball is thrown from the foot of a plane inclined at an angle θ to the horizon. If after striking the plane at a distance l from the point of projection it rebounds and retraces its former path, show that the velocity of projection is

$$\sqrt{\left\{\frac{(1+3\sin^2\theta)\,gl}{2\sin\theta}\right\}}. \qquad [\text{ C. H. } 1966]$$

42. A, B, C are the masses of three perfectly elastic balls moving in a straight line in the same sense with velocities a, b, c respectively. A impinges upon B and then B upon C, so that their velocities after impact are u, v, w respectively, v being the velocity of B after impact on C. Prove that

$$Au + Bv + Cw = Aa + Bb + Cc$$

$$Au^2 + Bv^2 + Cw^2 = Aa^2 + Bb^2 + Cc^2. \qquad [\text{ C. H. } 1908 \text{ }]$$

ANSWERS

1. 64 ft. **2.** $\frac{1}{8}$. **5.** 45 ft.

7. $16\sqrt{3}$ ft. per sec. at an angle 30° with the plane; 17·3 units nearly. **8.** $10\sqrt{10}$ ft. per sec.; at $\tan^{-1}\frac{1}{3}$ with the plane.

11. 10 ft. **12.** $4\frac{2}{3}$ and $5\frac{2}{3}$ ft. per sec.; $6\frac{2}{3}$ units.

14. 80 cms. **16.** $2/(\lambda - 1)$; $\lambda \nless 3$. **18.** $\dfrac{w}{P-w}\left(\dfrac{W}{W+w}\right)^2 \cdot (1+e)^2 x$.

19. 4·16 seconds; 1·93 seconds. **21.** $\frac{8}{27}$ ft.-lbs. **22.** $\frac{2}{3}$.

25. $8\frac{1}{3}$ ft.-lbs. **27.** Velocity of the nth ball is $\left(\dfrac{1+e}{2}\right)^{n-1} \cdot v$.

28. $3\sqrt{3}$ ft. per sec. along the line of centres; $2\frac{1}{4}$ ft.-lbs.

29. 4·62 cms. per sec. at 60° to the line of centres; 4·16 cms. per sec. at $\tan^{-1}\dfrac{\sqrt{3}}{7}$ to the line of centres; $\dfrac{16}{3}\sqrt{3}$ units.

36. $\frac{1}{3}$; $16\sqrt{6}$ ft. per sec. **38.** 100 lbs. wt.

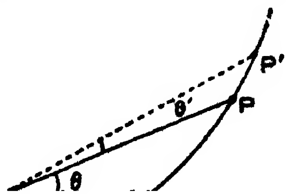
CHAPTER XIII

ANGULAR VELOCITY

131. Angular Velocity.

The angular velocity of a point moving on a plane, about any assumed point on that plane, is the rate of change of the angle made by the line joining the point to the moving point with any straight line drawn in a fixed direction in that plane.

If, for instance, P be a moving point on a plane and O be any point in that plane about which the angular velocity is required, then Ox being a fixed straight line through O, the rate at which the angle xOP $(=\theta)$ chages, as P traces out its path on the plane, is defined to be the angular velocity of P about O.



We may also define *the angular velocity of P about O as the rate at which the straight line OP turns about O with the motion of P.*

Uniform angular velocity.

The angular velocity of a moving point P about a given point O is said to be uniform when the straight line OP turns in the same plane through equal angles about O in equal times, however small the time intervals may be taken.

In case of uniform angular velocity, it may be measured

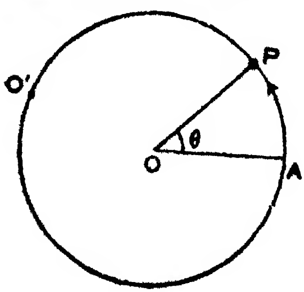
by considering the total angle through which the line OP turns about O in any time and dividing it by the time.

In case the angular velocity is not uniform, the angular velocity of P about O at any instant is measured by the limiting value of the ratio $\frac{\theta'}{t'}$, where θ' is the angle POP' described by OP in an infinitely small interval t' from the instant in question.

182. Uniform circular motion.

If a point moves in a circle of radius r with a uniform speed v, its angular velocity about the centre is uniform, and equal to $\frac{v}{r}$.

Let the moving point P describe a circle with centre O and radius r, with a uniform speed v. As the speed is uniform, the point traces out equal arcs of its path in equal times, and as equal arcs of a circle subtend equal angles at the centre, it follows that the angles turned over by OP about O are equal in equal times and this is true however small these equal intervals of time may be taken. Hence, the angular velocity of P about O is uniform.



Again, in time t the arc traced over by P is vt and the angle θ subtended by it at the centre O is $\frac{vt}{r}$ in circular measure. Hence, ω denoting the uniform angular velocity of P about O (in radians per second),

$$\omega = \frac{\theta}{t} = \frac{vt}{r} \Big/ t = \frac{v}{r}.$$

Conversely, under the above circumstances if ω be given, we can get v from $\mathbf{v} = \omega \mathbf{r}$.

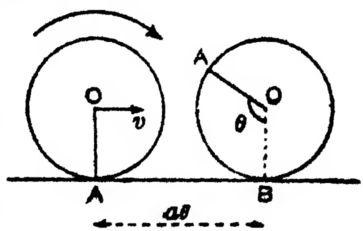
Cor. As equal arcs of a circle subtend equal angles at any point on the circumference, which are half the corresponding angles at the centre, it follows that *for a moving point P describing a circle of radius r with a uniform speed v, the angular velocity about any point O' on the circumference is uniform and equal to* $\frac{v}{2r}$.

About any other point on the plane, the angular velocity of the moving point in this case is clearly non-uniform.

13·3. A rolling wheel.

Let a wheel of radius a be rolling along a straight line on a rough ground without slipping, advancing with a uniform speed v.

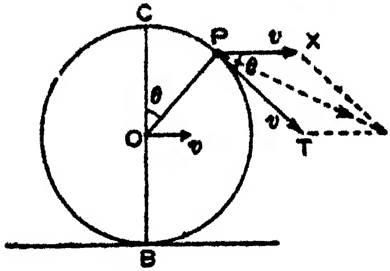
As there is no slipping in this case, in the time that the wheel makes one revolution about its centre, the distance advanced by the wheel or its centre, is equal to the circumference of the wheel. In any time the distance AB, advanced by the wheel is equal to the arc BA, described by any point A on its circumference round its centre. The motion of the rolling wheel therefore can be regarded as combination of a forward motion as a whole with a speed v as in case of slipping, together with a rotation of the wheel about its centre at the rate of one revolution in the



time $\frac{2\pi a}{v}$ in which the wheel moves forward through a distance equal to its circumference. The angular velocity of the wheel about its centre is thus,

$$\omega = 2\pi \Big/ \frac{2\pi a}{v} = \frac{v}{a}.$$

Thus, for a rolling wheel, any point P on the circumference at an angular distance θ from its topmost point C at any instant has a two-fold motion in space, one v in the forward horizontal direction PX in common with the centre, and the other



$\omega a = \dfrac{v}{a} \times a$, *i.e.*, v along the tangential direction PT due to the rotation round the centre. The *space velocity of the point P at the instant*, being a resultant of these two, *is equal to $2v \cos \dfrac{\theta}{2}$ in a direction bisecting the angle XPT.*

Cor. The *instantaneous velocity of the topmost point C* of the wheel (for which $\theta = 0$) *is* $2v$ in the forward horizontal direction.

The *lowest point B* of the wheel which at any instant is in contact with the ground (for which $\theta = \pi$) *has instantaneous velocity zero, i.e.*, it is instantaneously at rest. This point at any instant is defined as the **instantaneous centre of rotation** of the wheel. It may be noted that the motion of any point P of the wheel at any instant is the same as if it is rotating about B with an angular velocity $\dfrac{v}{a}$, so that the whole wheel, as it were, rotates instantaneously about B with the same angular velocity.

13·4. Illustrative Examples.

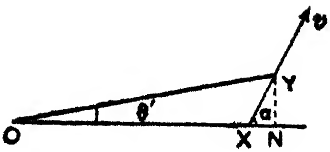
Ex. 1. *A particle at the point X has a velocity v in a direction making an angle α with the line OX. Show that the angular velocity of the particle at the instant about the point O is $\dfrac{v \sin \alpha}{OX}$.*

Let Y denote the position of the particle after an infinitely small time t' from the instant when it was at X. Then $XY = vt'$.

In this time, the angle through which OX turns about O is

$$XOY = \theta' \text{ say.}$$

Then YN being perpendicular on OX, $YN = XY \sin \alpha$ $= vt' \sin \alpha$. Also $YN = OY \sin \theta'$.



Thus, $vt' \sin \alpha = OY \sin \theta'$ $= OY . \theta'$ ultimately.

[Since, by Trigonometry, θ' being infinitely small, $\sin \theta' = \theta'$]

Now, by definition, *the angular velocity of the particle about* O is the ultimate value of $\frac{\theta'}{t'}$, when t' is infinitely small,

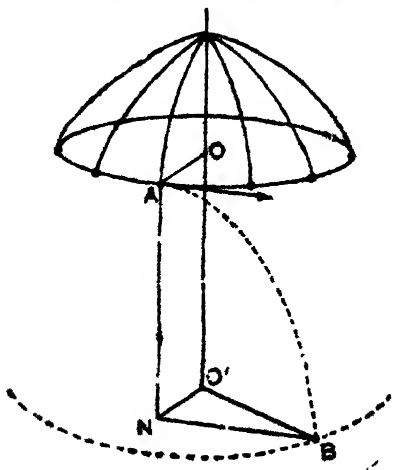
$$i.e., = \frac{\theta'}{t'} \text{ (in the limit)} = Lt \frac{v \sin \alpha}{OY} = \frac{v \sin \alpha}{OX}$$

$$= \frac{\text{component of } v \text{ perp. to } OX}{OX}$$

for ultimately, when t' is infinitely small, OY comes into coincidence with OX.

Ex. 2. *Drops of water are thrown tangentially off the horizontal rim of a rotating wet umbrella. The rim is 3 ft. in diameter and is held 4 ft. above the ground, and makes 14 revolutions in 33 seconds. Show that the drops of water will meet the ground on a circle of 5 ft. in diameter.*

O is the centre of horizontal circular rim of the umbrella, O' the point of the ground vertically below O, so that $OO' = 4$ ft.



As the umbrella makes 14 revolutions, *i.e.*, describes $14 \times 2\pi$ radians in 33 seconds, the angular velocity of any point of the rim is $\frac{28\pi}{33}$ and the radius being $\frac{3}{2}$ ft., the linear velocity is

$$\frac{28\pi}{33} \times \frac{3}{2} = \frac{28}{33} \times \frac{22}{7} \times \frac{3}{2} = 4 \text{ ft./sec.}$$

The drop from a point A of the rim is then thrown off with a horizontal velocity

4 ft./sec. in a vertical plane perpendicular to OA, and describing a parabolic orbit, falls on the ground at B.

Now, t denoting the time from A to B,

$$\tfrac{1}{2}gt^2 = \text{the vertical depth descended} = 4.$$

$$\therefore \qquad t^2 = 8/g = \tfrac{1}{4}, \text{ or } t = \tfrac{1}{2} \text{ sec.}$$

Also, the horizontal distance NB, (where AN is vertical) described by the particle $= 4t = 4 \times \frac{1}{2} = 2$ ft.

Thus, since OA or ON is perpendicular to the vertical plane of motion ANB,

$$BO = \sqrt{ON^2 + NB^2} = \sqrt{OA^2 + NB^2} = \sqrt{(\tfrac{3}{2})^2 + 2^2} = \tfrac{5}{2} \text{ ft}$$

Now, O being a fixed point, the locus of B is a circle of radius $\frac{5}{2}$ ft., *i.e.*, diameter 5 ft.

Examples on Chapter XIII

1. If the velocity of the extremity of the minute-hand of a clock is 20 times that of the extremity of the hour-hand which is 3 inches long, find the length of the minute-hand.

2. Find the velocity of any observer on the equator, taking the radius of the earth to be 4000 miles.

3. A boy is riding a tricycle along a road, the hind-wheels of the cycle being equal and of diameter $1\frac{3}{4}$ ft. If their angular velocity about their respective centres be 4π radians per sec., find the velocity of the cycle.

If the front wheel be of diameter 1 ft., find its angular velocity about the centre.

4. A boy runs at the rate of 5 miles per hour on the circumference of a horizontal wheel rotating about the vertical axis through its centre, and keeps the same position in space. Find the angular velocity of the wheel about its centre, assuming its radius to be $3\frac{2}{3}$ ft.

5. A wheel with its plane vertical is rolling on the ground, making 14 revolutions per minute. If the diameter of the wheel be 4 ft., find the space velocity of the points of the wheel at a height of 3 ft. above the ground.

6. Show that the velocity of the highest point of a wheel rolling on the ground is twice that of a point on the rim whose distance from the ground is half the radius.

7. A wheel rolls uniformly on the ground without sliding, its centre describing a straight line. Show that its angular velocity about the point of contact of the wheel with the ground is equal to the angular velocity of the wheel about its centre.

8. Two points describe the same circle in such a manner that the line joining them always passes through a fixed point. Show that at any instant their velocities are proportional to their distances from the point.

9. If two points P and Q are moving with velocities u, v making angles α, β respectively with the line PQ, then

$$\frac{u \sin \alpha - v \sin \beta}{PQ}$$

is the angular velocity of P relative to Q.

10. If two points describe the same circle of radius a in the same direction with the same speed u, show that at any instant their relative angular velocity is u/a.

11. A point moves uniformly along a straight line; show that its angular velocity about any point varies inversely as the square of its distance from that point.

12. Two small marbles P and Q are moving in a clockwise direction in concentric circular grooves of 3 inches and 4 inches radii respectively, on a smooth horizontal table, their respective velocities being 6 inches per second and 16 inches per second. If at any given instant they are 1 inch apart, find what time will elapse when they are 7 inches apart.

13. (i) Two points describe concentric circles in the same sense, with velocities varying inversely as the square roots of the radii of the circles. Find the angle subtended at the common centre by the line joining them when the relative angular velocity of one about the other vanishes.

(ii) Two points describe concentric circles of radii a and b with speeds varying inversely as the radii and with the same sense of rotation; show that the relative velocity

is parallel to the line joining these points when the angle θ between the radii to these points is given by

$$\cos\theta = \frac{2ab}{a^2+b^2}. \qquad [\text{ C. U. 1966 }]$$

14. If a point moves so that its angular velocity about two fixed points is the same, prove that it describes a circle.

15. A rod OA is rotating about the extremity O with angular velocity ω, and carries a rod AB which is rotating about A with angular velocity ω'. Show that the magnitude of the absolute velocity of the point B at any moment is

$$(a^2\omega^2 - 2ab\omega\omega'\cos\theta + b^2\omega'^2)^{\frac{1}{2}},$$

where $OA = a$, $AB = b$ and $\angle OAB = \theta$. [*C. U. 1942*]

16. A point P describes a circle of radius a, centre O, with uniform angular velocity ω; show that a point Q which describes a diameter AOB of the circle so that PQ is always perpendicular to AOB, moves from the mid-point of OA to the mid-point of OB in time $\pi/3\omega$. [*C. U. 1945*]

17. Two points P and Q are describing concentric circles of radii a and b with angular velocities ω and ω' respectively. Prove that the relative angular velocity of P with respect to Q when the distance between them is r, is

$$\{(r^2+a^2-b^2)\omega + (r^2+b^2-a^2)\omega'\}/2r^2. \qquad [\text{ C. H. 1964 }]$$

18. A wet open umbrella is held upright with its rim of radius a at a height h above the ground, and is rotated about the handle with uniform angular velocity ω. Show that the drops of water which fly off from the rim, will, on reaching the ground, be on a circle of radius

$$a\left(1 + \frac{2\omega^2 h}{g}\right)^{\frac{1}{2}}.$$

ANSWERS

1. 5 inches. 2. 1047 m.p.h.

3. $7\frac{1}{2}$ m.p.h. ; 7π radians per sec.

4. 2 radians/sec. 5. $2\sqrt{3}$ m.p.h. at $\pm 30^\circ$ to the horizon.

12. $1\frac{1}{7}$ sec.

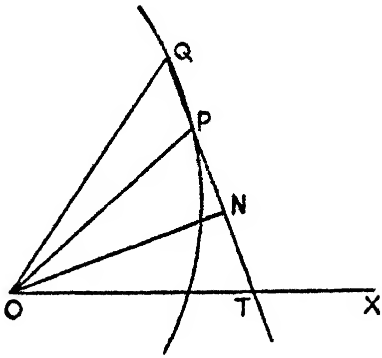
13. $\cos^{-1}\left(\dfrac{\sqrt{ab}}{a-\sqrt{ab}+b}\right)$, where a and b are the circles.

13·5. Analytical expression for Angular velocity.

Suppose a particle is moving along a curve and let P and Q be its positions on the curve at times t, $t+\Delta t$.

Let O be a fixed point and OX a fixed straight line.

Let $\angle XOP=\theta$, $\angle XOQ=\theta+\Delta\theta$, $OP=r$, $OQ=r+\Delta r$; let ON be perpendicular from O on PQ and let v be the velocity of the particle at time t.



Since the angular velocity ω of the point P about the point O is the rate of change of the angle θ, its analytical expression is given by

$$\omega=\frac{d\theta}{dt}. \qquad \dots \qquad \dots \qquad (1)$$

Now, $OP.OQ \sin POQ=2\triangle POQ=PQ.ON$,

$$\text{i.e., } r(r+\Delta r)\sin\Delta\theta=2\triangle POQ=\text{arc } PQ\cdot\frac{\text{chord } PQ}{\text{arc } PQ}\cdot ON.$$

Dividing by Δt, we have

$$r^2\frac{\sin\Delta\theta}{\Delta\theta}\cdot\frac{\Delta\theta}{\Delta t}+r\frac{\Delta r}{\Delta t}\sin\Delta\theta=2\frac{\Delta POQ}{\Delta t}$$

$$=\frac{\text{arc } PQ}{\Delta t}\cdot\frac{\text{chord } PQ}{\text{arc } PQ}\, ON.$$

Let $\Delta t \to 0$, then $\Delta\theta \to 0$ and $\sin \Delta\theta \to 0$ and $\frac{\text{chord } PQ}{\text{arc } PQ} \to 1$ and $\frac{\text{arc } PQ}{\Delta t} \to v$ and PQ becomes the tangent at P and ON becomes the perpendicular from O on the tangent P usually denoted by p and $\frac{POQ}{\Delta t}$ becomes the rate of the description of the area POQ, which is usually denoted by h.

$$\therefore \quad r^2 \frac{d\theta}{dt} = h = pv. \qquad \ldots \qquad \ldots \qquad (2)$$

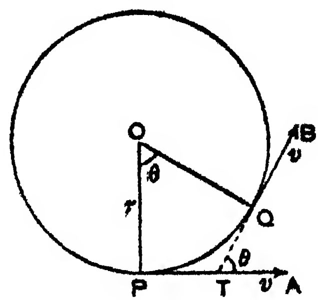
$$\therefore \quad r^2 \omega = pv. \qquad \ldots \qquad \ldots \qquad (3)$$

CHAPTER XIV

NORMAL ACCELERATION

14·1. *A particle describes a circle of radius r with a uniform speed v; to show that at any instant its acceleration is directed towards the centre, and is of magnitude* $\frac{v^2}{r}$.

Let a particle be moving along a circle of centre O and radius r with a uniform speed v.



At any point P of its path, its velocity is v along the tangent PTA. After an infinitely small time τ, the position of the particle being Q, its velocity is v along the tangent TQB at Q.

Let $\angle POQ$ be θ radians, then $\angle QTA$ is also θ radians, ($\because$ O, P, T, Q are concyclic) and so the velocity at Q may be broken up into components $v \cos \theta$ along PT and $v \sin \theta$ perpendicular to it, *i.e.*, parallel to PO.

But τ being infinitely small, θ is also infinitely small, and therefore, (as we know from Trigonometry), $\cos \theta = 1$, and $\sin \theta = \theta$ ultimately (in circular measure).

Hence, the velocity at Q is ultimately equivalent to the components v along PT and $v\theta$ parallel to PO.

Thus, in an infinitely small time τ from P, the change of velocity is zero along PT, and $v\theta$ parallel to PO. Hence, there is no acceleration along PT, and the only acceleration of the particle at P is $\frac{v\theta}{\tau}$ (in the limit) along PO.

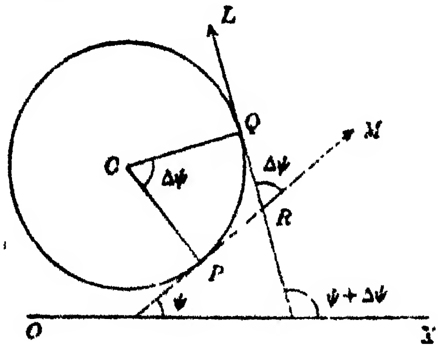
Now, θ (in circular measure) $= \frac{\text{arc } PQ}{r} = \frac{v\tau}{r}$.

Therefore, the resultant *acceleration of the particle at P*, which is directed *along the normal PO* is

$$\frac{v}{r} \times \frac{v\tau}{r} = \frac{v^2}{r}.$$

Analytically :

Let a particle be moving along a circle of centre C and radius r with a uniform speed v.



Let P be the position of the particle at time t and let Q be its position at time $t+\Delta t$ *i.e.*, after a short interval of time Δt. Let the tangents PM, QL at P and Q respectively make angles ψ, $\psi+\Delta\psi$ with the x-axis OX.

$\therefore$ $\angle PCQ = \angle QRM = \Delta\psi$ and from a property of the circle, it follows, $\Delta\psi/\text{arc } PQ = 1/r$. ... (1)

The velocity of the particle at P is v along PM. Since the particle is moving with uniform speed v, the velocity at Q is also v along the tangent QL and its component along the tangent PM is $v \cos \Delta\psi$ and that perpendicular to PM *i.e.*, along the normal PC is $v \sin \Delta\psi$.

Let f_s and f_n be the acceleration components of the particle along the tangent and normal.

$$f_s = \underset{\Delta t \to 0}{Lt} \frac{v \cos \Delta\psi - v}{\Delta t} = -v \cdot \underset{\Delta t \to 0}{Lt} \frac{1 - \cos \Delta\psi}{\Delta t}$$

$$= -v . \underset{\Delta t \to 0}{Lt} \frac{2 \sin^2 \frac{1}{2}\Delta\psi}{\Delta t}$$

$$= -v . \underset{\Delta t \to 0}{Lt} \left(\frac{\sin \frac{1}{2}\Delta\psi}{\frac{1}{2}\Delta\psi}\right)^2 . \frac{\Delta\psi}{\Delta t} . \tfrac{1}{2}\Delta\psi.$$

When $\Delta t \to 0$, $\Delta\psi \to 0$ and so $Lt \dfrac{\sin \frac{1}{2}\Delta\psi}{\frac{1}{2}\Delta\psi} = 1$.

$$\therefore \quad f_s = -v . 1 \frac{d\psi}{dt} . 0 = 0. \qquad \dots \qquad \dots \qquad (2)$$

Thus *the acceleration along the tangent is zero.*

$$f_n = \underset{\Delta t \to 0}{Lt} \frac{v \sin \Delta\psi}{\Delta t}$$

$$= v . \underset{\Delta t \to 0}{Lt} \frac{\sin \Delta\psi}{\Delta\psi} . \frac{\Delta\psi}{\text{arc } PQ} . \frac{\text{arc } PQ}{\text{chord } PQ} . \frac{\text{chord } PQ}{\Delta t} .$$

From (1), $\dfrac{\Delta\psi}{\text{arc } PQ} = \dfrac{1}{r}$, and when $\Delta t \to 0$, $\Delta\psi \to 0$

and $Q \to P$, so $Lt \dfrac{\sin \Delta\psi}{\Delta\psi} = 1$, $Lt \dfrac{\text{arc } PQ}{\text{chord } PQ} = 1$

and $Lt \dfrac{\text{chord } PQ}{\Delta t} = v$.

[See *Authors' Differential Calculus, Appendix, Section D, Art. 7.*]

$$\therefore \quad f_n = v . 1 . \frac{1}{r} . 1 . v = \frac{v^2}{r} . \qquad \dots \qquad (3)$$

From (2) and (3), it follows that the resultant acceleration of the particle at P is $\dfrac{v^2}{r}$ and it is along the normal PO and hence directed towards the centre.

Cor. *If ω be the angular velocity about the centre of a particle moving uniformly in a circle of radius r, the linear speed being $v=\omega r$, its normal acceleration is $\omega^2 r^2/r=\omega^2 r$.*

Note. Even when the particle describes the circle with a *non-uniform speed*, if v be its speed at any point P, the acceleration at that point along the normal PO will still be v^2/r, but in this case there will also be a tangential component of acceleration $\frac{dv}{dt}$ which is not zero.

14·2. Centripetal and Centrifugal Forces.

We have seen above, that when a body (of mass m say) moves in a circle of radius r with a speed v, it has an acceleration v^2/r which at any instant is directed towards the centre. Necessarily therefore, *there must be a force $m(v^2/r)$ towards the centre acting on the body in order that it may move in a circle. This force is known as* **Centripetal force.**

For example, we can make a stone move in a circle by attaching it to one end of a string, and whirling it round with the hand by the other end. In this case, the tension T of the string is the necessary centripetal force on the stone, and we must have $T=m\,(v^2/r)$, where m is the mass of the stone, v its speed, and r the length of the string.

In this case again, there is an equal and opposite reaction or a counterpull on the hand, and there is a feeling as if the stone pulls the hand with a force, trying to fly away from the centre along the length of the string. This outward force which a body moving in a circle appears to exert at the centre, and which is really a force *equal and opposite to the centripetal force, is known as the* **Centrifugal force.**

It may however be noted that the body really does not tend to fly away from the centre along the radius, for if the string be cut, it flies off along the tangent line.

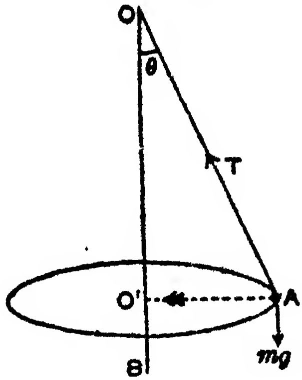
Several other examples of normal acceleration and centripetal force are appended below.

14·3. The Conical Pendulum.

If a heavy particle A be tied to one extremity of a string and the other extremity be attached to a fixed point O,

and the system be rotated uniformly about a vertical line OB through O, a steady state of motion will be reached with the string making a definite angle to the vertical, and describing a cone. The particle accordingly will describe a horizontal circle with centre O' as shown in the figure. The system constitutes what is called a Conical Pendulum.

We can easily deduce a relation between the angular velocity ω with which the string or the particle rotates



and the inclination of the string to the vertical. For, if l be the length of the string, the radius of the circle described by A is $l \sin\theta$. Hence, the normal acceleration of A is $\omega^2 l \sin\theta$ along AO'. If m be the mass of the particle, the centripetal force required to make the particle move in the above circle is $m\omega^2 l \sin\theta$, and this is supplied by the component of the tension T of the string along the radial line AO'.

Thus, $T \sin\theta = m\omega^2 l \sin\theta$ ... ... (i)

whence $T = m\omega^2 l$ ... ... (ii)

(in absolute units, if ω be in radians per sec.)

Again, since there is no vertical motion of the particle, the forces in the vertical direction must balance.

Hence, $T \cos\theta = mg$. ... ... (iii)

$$\cos\theta = \frac{mg}{T} = \frac{g}{\omega^2 l}, \quad \text{or,} \quad \theta = \cos^{-1}\frac{g}{\omega^2 l} \quad \cdots \text{(iv)}$$

Lastly, since $\omega = \sqrt{\dfrac{g}{l\cos\theta}} = \sqrt{\dfrac{g}{OO'}}$, the period of revolution of the particle A is evidently

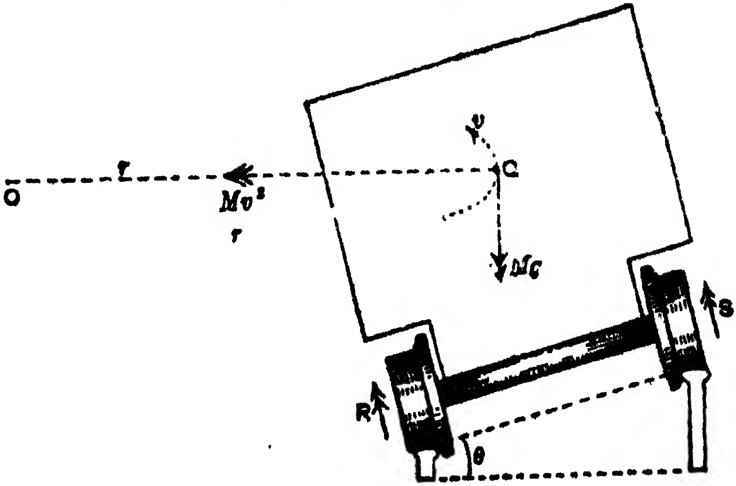
$$\frac{2\pi}{\omega} = 2\pi\sqrt{\frac{OO'}{g}}, \quad i.e., \quad \propto \sqrt{OO'}.$$

Note. If $\omega^2 l < g$, or, $\omega < \sqrt{\dfrac{g}{l}}$, the value of θ given by (iv) is impossible. In this case, from (i), $\sin\theta = 0$, or, $\theta = 0$ and from (iii), $T = mg$. The string accordingly will remain vertical, though the system may be rotating, so long as $\omega < \sqrt{\dfrac{g}{l}}$. For greater angular speed, the particle will fly off and the string becomes inclined to the vertical.

14·4. Motion of a railway carriage on a curved portion of a railway line.

In order that a railway carriage moving on the rails may not slip off the rails, the wheels of the carriage have got flanges on one side (usually on the inner side of both pair of wheels) so as not to allow the wheels to move sideways, one way or the other. When the carriage is taking a bend, if both the rails be at the same level, the centrifugal tendency for carriage to fly off the rails outwards is prevented by the flanges of the wheels pressing against the rails, the reaction supplying the necessary force towards the centre of the bend for the motion on the curved path. This would produce a huge amount of friction between the flanges and the rails, sufficient to wear out the flanges quickly. In order to avoid this, the outer rail of the bend is generally raised a little, so that the floor of the carriage moving on the rails is not horizontal but inclined. The inclination is calculated so as to reduce the friction between the flanges and the rails to nil, and depends on the amount of curvature of the path, as also on the average speed with which the trains would move at the bend.

The necessary inclination may be calculated as follows :



Let θ be the inclination to the horizon of the line joining the top of the rails, r the radius of the bend taken by the carriage, v its speed there.

Let R and S be the reactions of the rails exerted normally on the wheels, which, as is apparent from the figure, are upwards at an inclination θ to the vertical. We assume that there is no sideways reaction between the flanges of the wheels and the rails.

The vertical components of R and S balance the weight of the carriage, and the horizontal components produce the necessary normal acceleration $\frac{v^2}{r}$ towards the centre of the curvature for the motion of the carriage along the curved path

Hence, M being the mass of the carriage,

$$(R+S)\cos\theta = Mg$$

$$(R+S)\sin\theta = M\frac{v^2}{r}$$

giving, $\tan\theta = \frac{v^2}{gr}$, or, $\theta = \tan^{-1}\frac{v^2}{gr}$.

If d be the distance between the rails, the excess of the height of the outer rail over the inner one, at the bends, is

$$x = d\tan\theta = \frac{v^2 d}{gr}.$$

Note 1. In case a train runs with a quicker or a slower speed than that for which the elevation is calculated at a bend, sideways pressures, one way or the other, will be produced between the flanges and the rails, in addition to the normal reactions R and S, and components of these should also be taken into account in writing the equations given there.

Note 2. 'd' is called by the engineers as 'Railway Cant'.

145. Motion of a bicycle rider.

A man is riding a cycle with uniform speed v round a curved path of radius r; to find the angle at which the cycle must be inclined to the vertical.

Let m be the mass of the man and the cycle. Since the cycle is moving along a circular path, there is a force mv^2/r towards the centre of the circle and hence the reaction of the ground is not vertical but is inclined to the vertical. Consequently the rider inclines his body and the cycle *inwards* towards the centre of the circular path.

Let G be the combined C.G. of the cycle with the rider, and P the point of contact of the cycle with the ground. Let θ be the constant inclination of the cycle with the vertical (*i.e.*, the angle which PG makes with vertical). Since the cycle has no angular motion in the vertical plane through PG, the sum of the moments of all the forces acting in this plane on the system about G is zero*. Hence, if R be the total reaction of the ground at P in the above plane, the moment of R about G is zero, *i.e.*, R must be along PG (the only other force passing through G).

*The proof depends on Dynamics of rigid bodies, and is beyond the scope of this book.

Now, the horizontal component of R supplies the force necessary to produce the acceleration towards the centre.

$$\therefore \quad R \sin \theta = m\,(v^2/r). \qquad \cdots \quad (1)$$

Also the vertical component of the reaction balances the weight of the man and the cycle.

$$\therefore \quad M \cos \theta = mg. \qquad \cdots \quad (2)$$

From (1) and (2), by division, we get $\tan \theta = v^2/rg$.

$\therefore$ the cycle must be inclined to the vertical at an angle $\tan^{-1}(v^2/rg)$.

Note. It is a familiar fact that when a cyclist moving very rapidly on a level path rounds a corner, he often leans *inwards*.

14·6. Hodograph.

—If P be the position of a particle moving in any manner, and if from a fixed origin O, a line OQ is drawn to represent in magnitude, direction, and sense, the velocity of P, the locus of Q is called the *hodograph* of the path of P.

O is called the *pole* of the hodograph.

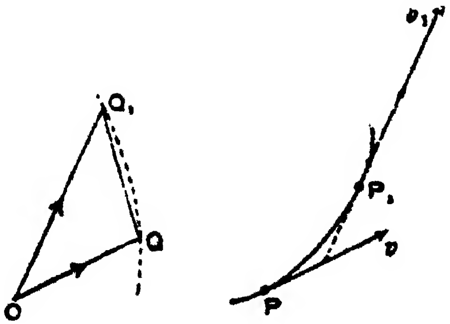
Suppose, $P, P_1, P_2, P_3, \ldots\ldots$ are consecutive positions of the particle in its path, $v, v_1, v_2, v_3, \ldots$ are the velocities at those points, and if from a fixed point O, lines $OQ, OQ_1, OQ_2, OQ_3, \ldots$ are drawn to represent the respective velocities in magnitude, direction and sense, then $Q, Q_1, Q_2, Q_3, \ldots$ will lie on a curve which is called the hodograph of the path of P.

It should be noted that $OQ, OQ_1, OQ_2, OQ_3, \ldots$ are parallel to the tangents at $P, P_1, P_2, P_3, \ldots$

The points $Q, Q_1, Q_2, Q_3, \ldots$ are said to correspond to the points $P, P_1, P_2, P_3, \ldots$

Theorem. *If the hodograph of the path of a moving particle P be drawn, then at any instant, the velocity of the corresponding point Q in the hodograph represents in*

magnitude, direction and sense, the acceleration of the point P in its path.



Let P, P_1 be any two consecutive positions of the moving particle on its path, and let Q, Q_1 be the corresponding points on the hodograph with respect to O as pole.

Now, OQ, OQ_1 represent in magnitude, direction and sense, the velocities of the particle at P, P_1 respectively.

Suppose, the particle moves from P to P_1 in a very small interval of time t.

Since, by the triangle of velocities,

$$\text{velo. } OQ_1 = \text{velo. } OQ + \text{velo. } QQ_1.$$

$\therefore$ the change of velo. of P in time t is QQ_1.

$\therefore$ the acceleration of the particle P

$$= \frac{QQ_1}{t}, \text{ when } t \text{ is made infinitely small.} \quad \cdots \quad (1)$$

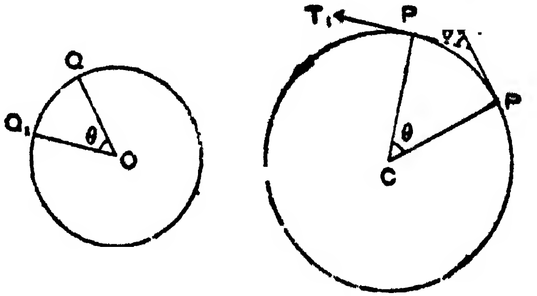
Also, during the same time, Q moves to Q_1 on the hodograph.

$\therefore$ velo. of Q is equal to

$$\frac{\text{arc } QQ_1}{t} \text{ when } t \text{ is made infinitely small.} \quad \cdots \quad (2)$$

Since, when t is infinitely small, the arc QQ_1 is ultimately equal to the chord QQ_1, it follows from (1) and (2) that the acceleration of the particle P is equal to the velocity of Q on the hodograph.

14·7. If a particle is describing a circle of radius r with a uniform speed v, its acceleration at any moment is $\frac{v^2}{r}$ directed towards the centre. (*Alternative Proof*)



Let P, P_1 be two consecutive positions of the particle, and Q, Q_1 be the corresponding points on the hodograph.

Since $OQ = v =$ a constant, therefore, the locus of Q, *i.e.*, the hodograph of the path of the particle P is a circle.

Suppose, the arc PP_1 is described in an infinitely small interval of time t.

Since OQ, OQ_1 are parallel to the tangents at P, P_1,

$$\therefore \quad \angle QOQ_1 = \text{angle between tangents at } P \text{ and } P_1$$
$$= \angle PCP_1.$$

$$\therefore \quad \frac{\text{arc } QQ_1}{OQ} = \frac{\text{arc } PP_1}{CP}.$$

$$\therefore \quad \frac{\text{arc } QQ_1}{t} = \frac{OQ}{CP} \cdot \frac{\text{arc } PP_1}{t}.$$

$$\therefore \quad \text{velo. of } Q \text{ on the hodograph} = \frac{v}{r} \cdot v = \frac{v^2}{r}.$$

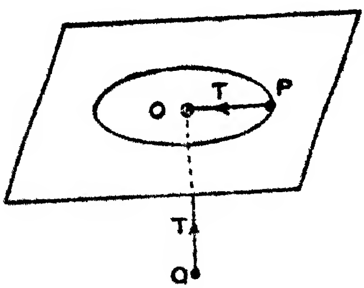
Since the velocity of Q is along QQ_1, which is ultimately the tangent at Q, and hence perp. to OQ, *i.e.*, perp. to PT, and so parallel to PC, and since the sense from Q to Q_1 corresponds to the sense from P to O, and as we know that the acceleration of P on the circle is equal to the velocity of Q in magnitude, direction and sense,

$\therefore$ the accel. of P on the circle $= \dfrac{v^2}{r}$ directed towards the centre.

14·8. Illustrative Examples.

Ex. 1. *Two masses P and Q are joined by a light inextensible string. The mass P describes a circle of radius 15 ft. on a smooth horizontal table with a uniform speed, while Q is suspended vertically in equilibrium by the string which passes through a small hole in the table at the centre of the circle described by P. If the masses of P and Q are 96 and 125 lbs. respectively, find the speed of P.* ($g = 32$ *ft./sec*^{2}) [*C. U. 1934*]

Let v denote the speed of P, so that its acceleration towards the centre of the circle is $\dfrac{v^2}{15}$, and hence, T being tension of the string, considering the motion of P,



$$T = 96 \times \frac{v^2}{15} = \frac{32v^2}{5} \quad \dots \text{(i)}$$

Again, Q remaining fixed in space, its weight is balanced by the tension of the string so that

$$T = 125g = 125 \times 32 \dots \text{(ii)}$$

From (i) and (ii),

$$\frac{32v^2}{5} = 125 \times 32, \quad \text{or,} \quad v^2 = 625.$$

$\therefore$ $v = 25$ ft./sec.

Ex. 2. *The maximum weight a string can support is 121 lbs. A mass of 48 lbs. is suspended from one end of it, and the other extremity*

is attached to the top of a vertical rod. The rod is made to rotate about itself. If the length of the string is 6 ft., find the maximum number of revolutions the rod can make per minute without breaking the string and the maximum inclination of the string to the vertical.

Let n denote the number of revolutions per minute and θ the inclination of the string to the vertical when the motion is steady, T the tension in the string. The angular velocity of the system is then $\frac{n \times 2\pi}{60}$ radians per second.

Now, with the figure of § 14·3 (since $m=48$ lbs. and $OA=6$ ft.), we get

$$T \sin \theta = 48 \left(\frac{n \times 2\pi}{60}\right)^2 \times 6 \sin \theta \quad \cdots \quad \text{(i)}$$

$$T \cos \theta = 48g. \quad \cdots \quad \cdots \quad \text{(ii)}$$

From (i),

$$\therefore \quad n = \frac{60}{2\pi}\sqrt{\frac{T}{6 \times 48}}$$

and since the maximum possible value of $T = 121$ lbs. wt. $= 121 \times 32$ poundals, the greatest value of n is

$$\frac{30 \times 7}{22}\sqrt{\frac{121 \times 32}{6 \times 48}} = \frac{15 \times 7}{11} \times \frac{11}{3}$$

$$= 35 \text{ revolutions per minute.}$$

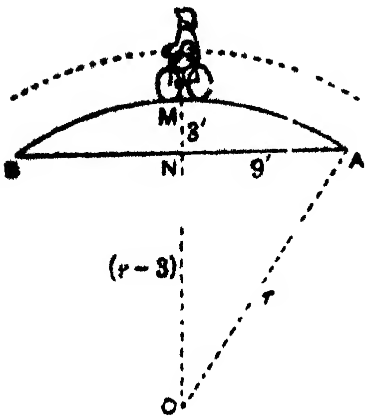
For this case, from (ii), the least value of

$$\cos \theta = \frac{48g}{121g} = \frac{48}{121},$$

i.e., the greatest value of $\theta = \cos^{-1} \frac{48}{121}$.

Ex. 3. *On a ditch of breadth 18 ft. there is a bridge in the form of a circular arc, the middle point of which is at a height 3 ft. from either extremity. Find the greatest speed with which a cyclist can pass over the bridge with safety if the height of the combined centre of gravity of the man and the cycle is 3 ft. above the point of contact of the wheels with the ground.*

AB, the breadth of the ditch, is 18, and the middle point M of the circular bridge is at a height 3 above AB, so that $MN = 3'$ where N is the middle point of AB.



Thus, $AN = 9$, and O being the centre of the circular bridge, if r be its radius, from the triangle OAN,

$$r^2 = (r-3)^2 + 9^2$$

whence $r = 15$ ft.

As the cyclist is moving over the bridge, the C.G., being at a height 3 ft. above the point of contact, describes a circle of radius 18 ft. Hence, v being the speed of the cycle, m the combined mass moving, the centrifugal force is $m\frac{v^2}{18}$, whereas the downward weight is

$$mg = 32m.$$

Thus, if $\frac{mv^2}{18} > 32m$, i.e., $v^2 > 18 \times 32$,

or, $v > 24$ ft./sec., the cycle will tend to fly away and lose its contact with the ground, for the weight downwards (i.e., towards the centre) is insufficient to supply the necessary centripetal force for the circular motion.

Thus, the greatest speed with which the cyclist can pass over the bridge with safety is

$$24 \text{ ft./sec.} = 24 \times \frac{30}{44}, \text{ or, } 16\frac{4}{11} \text{ miles per hour}$$

Examples on Chapter XIV

1. A particle of mass 15 lbs. connected with a fixed point on a smooth horizontal plane by a string of length 9 ft. moves uniformly in a circle on the plane with a velocity of 6 ft. per sec. Find the tension of the string. [*C. U. 1936*]

2. If the Moon revolves about the Earth in a circle whose radius is 240000 miles, performing one revolution in 30 days, find its acceleration in foot-second units.

*3. A stone of mass 2 lbs. is attached at one end of a string 5 ft. long, the other end of which is fixed, and the stone moves in a horizontal circle. If the string can just bear a weight of 605 lbs., find the greatest number of revolutions per second that can be made without breaking the string.

4. A stone, held by a string of length 6 ft., describes a circle on a smooth horizontal table whose centre is the fixed end of the string. If the tension of the string be three times the weight of the stone, find the time of revolution.

5. A point X moves in a circle with uniform angular velocity ω. If C be the centre of the circle, and M the foot of the perpendicular from X on a fixed diameter, show that the acceleration of M is $\omega^2.MC$ towards C.

6. Two equal masses which are attached by inextensible strings to two fixed points are describing circles round them. If the times of revolution are the same, show that the tensions of the strings are proportional to their lengths.

7. The attraction exerted by the Sun on any of its planets varies directly as the mass of the planet, and inversely as the square of the distance of the planet from the Sun. Show that the square of the times of revolution of the planets vary as the cubes of the radii of the orbits (which are all supposed circle).

8. A string $OABC$, where $OA = AB = BC$, with masses each equal to m fastened at A, B, C rotates about O, on a smooth horizontal table with the string always remaining straight. Show that the tensions of the portions are as 6 : 5 : 3.

*9. A particle of mass m on a smooth horizontal table is fastened to one end of a fine string which passes through a small hole in the table, and supports at its other end a particle of mass $2m$, the particle m being held at a distance

a from the hole. Find the velocity with which m must be projected horizontally so as to describe a circle of radius a.
[*C. U. 1940*]

10. Two equal particles are connected by a string passing through a hole in a smooth horizontal table, one particle being on the table, the other hanging vertically. How many revolutions per minute would the particle on the table have to perform in a circle of radius 6 inches in order to keep the other particle at rest ?

11. A string whose length is l passes through a heavy smooth ring and has its ends attached to two points distant a apart, in the same vertical line. Show that when the ring rotates in a horizontal circle, the portion of the string between the ring and the lower point of the support will be horizontal if the angular velocity ω is given by

$$\omega^2 = 2g\frac{l^2}{a(l^2-a^2)}. \qquad [\ C.\ U.\ 1944\ ]$$

12. A mass of 5 lbs. rotates as a conical pendulum at the end of a string 5 ft. long, which can just sustain a weight of 20 lbs. Find the greatest number of complete revolutions that can be made by the string in one minute without breaking.

13. If the velocity of the bob of a conical pendulum is v, and the length l, show that θ, the inclination of the string to the vertical is given by

$$gl(1-\cos^2\theta)-v^2\cos\theta=0.$$

14. A mass of 8 lbs. is connected by a string of length 5 feet to a point 4 ft. above a smooth horizontal table. If the particle revolves with a velocity 3 ft. per sec. on the table, find the pressure on the table.

15. A heavy particle fastened by a light inextensible string to a fixed point O is moving in a horizontal circle at the rate of n revolutions per second. Prove that the point O is at a distance $g/4\pi^2n^2$ vertically above the centre of the circle.

16. A merry-go-round consists of a horizontal circle of radius 3 ft. revolving about a vertical axis through its centre

at the rate of 7 revolutions in 22 seconds. A boy of weight 56 lbs. is seated on a light wooden horse suspended by a string from the revolving circle. If the inclination of the string to the vertical be $\tan^{-1}\frac{3}{4}$, find the length of the string and also its tension.

17. A train is travelling at the rate of 40 miles an hour on a curve of radius 800 ft., on a narrow-gauge railway. If the gauge of the line is 3 ft., find how many inches the outer rail must be raised above the inner, so that there may be no lateral pressure on the rails.

18. A train running at a speed of 30 miles per hour is rounding a curve of radius 484 ft. Find what horizontal force will keep vertical the string by which a body of mass 8 lbs. hangs from the roof of a carriage rounding the curve.

19. At what angle must a cyclist incline his machine to the vertical so that he may keep himself on to a circular path of radius 121 ft. when running at a uniform speed of 7·5 miles per hour? (Take $g = 32$ ft./sec^2) [*C. U. 1950*]

20. A curve on a railway line is banked up so that the lateral thrust on the inner rail due to a truck moving with speed u_1 is equal to the thrust on the outer rail when the truck is moving with speed u_2 $(u_2 < u_1)$. Show that there will be no lateral thrust on either rail when the truck is moving with speed v, where,

$$v^2 = \tfrac{1}{2}(u_1^2 + u_2^2).$$

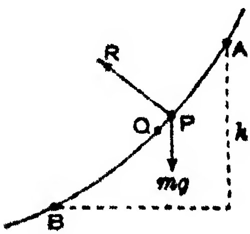
ANSWERS

1. 60 poundals.	2. ·0074 ft/sec^2.	3. 7.	4. $1\frac{4}{7}$ sec.
9. $\sqrt{2ag}$.	10. $76\frac{4}{11}$.	12. 48.	14. 7 lbs. wt.
16. 5 ft, 70 lbs. wt.	17. 4·8 inches.	18. 1 lb. wt.	19. $\tan^{-1}\frac{1}{32}$.

CHAPTER XV

MOTION ON A SMOOTH CURVE UNDER GRAVITY

15·1. We know from the principle of energy (§ *10·6*) that when a body moves under any force, the change in its kinetic energy in anytime is equal to the work done by the acting forces. Now, when a particle of mass m slides on a smooth curve under gravity, at any point P of its path, the only forces acting on it are its weight mg vertically downwards, and the reaction R of the smooth curve which is along the normal to the curve at the point P. As the point moves through an infinitesimal distance PQ along the curve, since this displacement is ultimately perpendicular to the direction of R, the work done by R is zero. As this is true at every point throughout the path of the particle, the sum-total of the work done by the normal reaction is zero throughout. Hence, when the particle moves from a point A to a point B of its path, the only work done by the acting forces is the work done by the weight mg which is constant in magnitude and direction, and if h be the vertical depth of B below A, this work done is mgh.



Thus, if u and v be the velocity of the particle at A and B respectively,

$$\tfrac{1}{2}mv^2 - \tfrac{1}{2}mu^2 = mgh,$$

or, $$v^2 - u^2 = 2gh, \quad i.e., \quad v^2 = u^2 + 2gh.$$

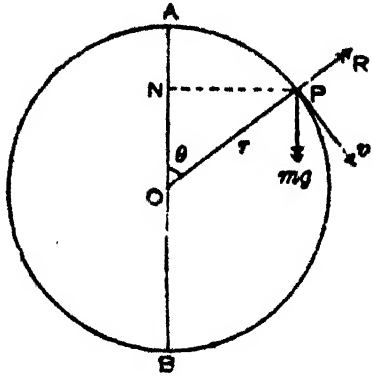
If the particle be sliding up, and B be above A, we get similarly

$$v^2 = u^2 - 2gh.$$

15·2. Motion in a vertical circle.

A. *A particle slides down the outside of the arc of a smooth vertical circle, starting from rest at the highest point ; to investigate its motion.*

Let O be the centre, A the highest point and r the radius of the circle.



Let P be the position of the particle on the circle at any instant, and v be its velocity there. Let m be the mass of the particle.

Draw PN perp. to OA and let $AN=h$ and $\angle AOP=\theta$.

The forces acting on the particle are

(i) the reaction R acting along the normal OP outwards ;

(ii) its weight mg vertically downwards.

$\therefore$ the total force along $PO = mg\cos\theta - R$.

Since the particle is describing a circle, the resultant force must be mv^2/r along PO [*Art. 14·1, Note*].

$$\therefore \quad mg\cos\theta - R = \frac{mv^2}{r}. \qquad \cdots \quad (1)$$

Again, since the body slides under the action of gravity on a smooth curve, we have by the Principle of Energy,

$$\tfrac{1}{2}mv^2 = mg.AN = mgh \qquad [\textit{ Art. 15·1 }]$$

$$= mg\,(OA - ON) = mgr\,(1-\cos\theta).$$

$$\therefore \quad v^2 = 2gh = 2gr\,(1-\cos\theta). \qquad \cdots \quad (2)$$

$\therefore$ from (1) and (2), we get

$$mg\cos\theta - R = m.\frac{2gr(1-\cos\theta)}{r} = 2mg(1-\cos\theta).$$

$$\therefore \quad R = mg\,(3\cos\theta - 2)$$

$$= mg\left(3\frac{r-h}{r} - 2\right) = \frac{mg}{r}(r-3h). \qquad \cdots \quad (3)$$

From (3), it is clear that as h increases, R decreases, and when $h = \frac{1}{3}r$, R vanishes. When $h > \frac{1}{3}r$, R becomes negative, which is impossible, since for the normal reaction can never be inwards.

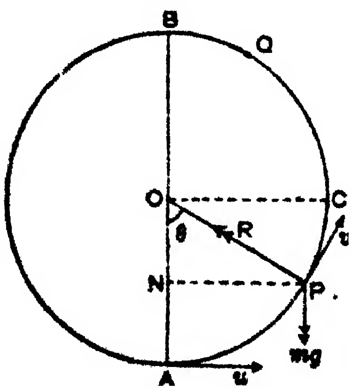
Hence, the *necessary condition for the motion of the particle along the outside of the circumference of the circle is $h \ngtr \frac{1}{3}r$.*

As soon as h *just* exceeds $\frac{1}{3}r$, the particle *leaves the curve*, which is the implication of R here being negative.

Note 1. When the particle leaves the curve, $v^2 = \frac{2}{3}gr$, *i.e.* $v = \sqrt{\frac{2}{3}gr}$ and $\cos\theta = \frac{2}{3}$.

Note 2. After leaving the circle at P, (where $\angle AOP = \cos^{-1}\frac{2}{3}$) the subsequent motion of the particle will be the same as that of a particle projected with velocity $v = \sqrt{\frac{2}{3}gr}$ in the downward direction of the tangent to the circle at P. In other words, *on leaving the circle, the particle will describe a parabolic path.*

B. *A particle is projected from the lowest point of a smooth vertical circle and moves along the inside of the circular arc ; to investigate its motion.*



Let r be the radius of the circle, O its centre, AOB the vertical diameter, and u the velocity of projection of the particle at the lowest point A.

Let v be the velocity of the particle at any point P on its path, and R be the reaction of the curve there. Draw PN perp. to AO and let $AN = h$ and let $\angle AOP = \theta$.

Then, by the principle of energy,

$$\tfrac{1}{2}mv^2 - \tfrac{1}{2}mu^2 = -mgh. \qquad [\textit{Art. 15·1}]$$

$$\therefore \quad v^2 = u^2 - 2gh$$
$$= u^2 - 2gr(1 - \cos\theta)$$
$$= u^2 - 2gr + 2gr\cos\theta. \quad \cdots \quad \text{(i)}$$

The forces acting on the particle at P are

(i) the weight mg ;

(ii) the reaction R of the smooth arc acting along the normal PO.

Since the particle is moving in a circle, the resultant force along the normal PO is mv^2/r. [*Art. 14·1, Note*]

$\therefore$ resolving the force along the normal PO, we have

$$R - mg\cos\theta = mv^2/r.$$

$$\therefore \quad R = mg\cos\theta + \frac{m}{r}\{u^2 - 2gr(1 - \cos\theta)\}$$

$$= \frac{m}{r}\{u^2 - 2gr + 3gr\cos\theta\}. \quad \cdots \quad \text{(ii)}$$

Equation (i) gives the velocity, and (ii) gives the reaction of the curve on the particle at any height.

As θ increases, $\cos\theta$ diminishes, and so v and R both diminish, their least possible values being given, when $\theta = 180°$, (*i.e.*, at the highest point B) by,

$$v^2 = u^2 - 4gr$$

and $$R = \frac{m}{r}(u^2 - 5gr).$$

Case I. If $u^2 > 5gr$,

both v and R remain positive, even at the highest point B, so that neither the motion of the particle stops, nor does it leave the circle anywhere.

The particle therefore makes complete revolutions along the circle.

Case II. If $u^2 \not> 2gr$,

from (i), $v = 0$ when $\cos\theta = \frac{2gr - u^2}{2gr}$, giving an acute-angled value for θ, and (ii) shows that $R \left(= mg \cos\theta + \frac{mv^2}{r} \right)$ is positive still. Thus, the particle comes to rest, without leaving its contact with the circle, at some point P below the horizontal diameter OC. It then slides down and retraces its path, and passing through A rises on the other side through an equal height.

Thus, *the particle in this case oscillates in an arc less than a semi-circle on either side of the lowest point A.*

If $u^2 = 2gr$, the arc of oscillation is a semi-circle.

Case III. If $u^2 > 2gr$ *but* $< 5gr$,

from (i) and (ii), both v and R remain positive till $\theta = 90°$. After this θ being obtuse, $\cos\theta$ is negative, and as $3gr\cos\theta$ is then less than $2gr\cos\theta$, R vanishes before v^2, at a point Q given by

$$\cos\theta = -\frac{u^2 - 2gr}{3gr},$$

which since the fraction is numerically less than unity (for $u^2 < 5gr$), gives a real, obtuse-angled value for θ.

The *particle leaves the circle* here (at some point Q between C and B, before reaching the highest point), and as its velocity, given from (i) by $v^2 = u^2 - 2gr - \frac{2}{3}(u^2 - 2gr) = \frac{1}{3}(u^2 - 2gr)$, is still positive, with this initial velocity along the tangent to the circle at Q, the particle *describes a free parabolic path.*

Note 1. *If a particle is hanging from a fixed point by a light inextensible string* and is projected with a certain horizontal velocity u, the motion is exactly the same as that of a particle moving inside a smooth vertical circle [*case B*]. We have only to substitute the tension T for the reaction R and the length of the string l for r the radius of the circle in the discussion of the case (B) above. Thus,

(i) if $u > \sqrt{5gl}$, the particle makes complete revolutions ;

(ii) if $u \leq \sqrt{2gl}$, the particle oscillates on either side of the lowest point ;

(iii) if $u > \sqrt{2gl}$ but $< \sqrt{5gl}$, tension vanishes somewhere in the upper half of the circle, the string becomes slack and *the particle describes a parabola freely so long as the string does not become tight again.*

Note 2. *In the case of the motion of a bead on a smooth vertical circular wire*, or that of a particle moving inside a smooth vertical circular tube, the bead or the particle keeps to the circular path, and no question of its leaving the circular path arises. In the case when $u > \sqrt{2gr}$ but $< \sqrt{5gr}$, the pressure vanishes somewhere in the upper half of the circle and it changes sign, so that the bead, instead of pressing the wire outwards, begins to press it inwards above that point.

Examples on Chapter XV

1. A ball of mass 4 lbs. connected with a fixed point by means of an inelastic string of length 8 ft. hangs vertically. If it is projected horizontally with a velocity of 32 ft. per sec., find the tension of the string when the ball has risen through a vertical distance of 12 ft.

2. A particle is projected along the inside of a smooth vertical circular ring of radius r from the lowest point with a velocity u. If the particle leaves the ring at an angular distance of 60° from the top, show that $u^2 = \frac{7}{2}gr$.

3. A boy of weight 20 lbs. is placed on a light cradle which is supported by two parallel ropes each 6 ft. long, and which is swinging through an angle of 60° on each side of the vertical. Find the tension in each rope when the cradle is (i) at its highest, and (ii) at its lowest point.

4. A heavy particle of weight W, attached to a fixed point by a light inextensible string, describes a circle in a vertical plane. The tension of the string has the values

mW and nW respectively when the particle is at the highest and lowest point in its path. Show that

$$n = m + 6.$$

5. A bead slides down a smooth vertical circular wire from rest at the highest point. Show that its velocity at any point varies as the chord of the arc of descent.

6. A particle connected by an inelastic string to a fixed point moves in a vertical circle. Show that the sum of the tensions of the string when the particle is at the opposite ends of a diameter is constant.

7. A ball of mass 8 lbs. oscillates through 180° on the inside of a smooth circular hoop of radius 6 ft. fixed in a vertical plane. If v be the speed at any point, prove that the pressure on the hoop at that point is $2v^2$.

8. The roadway of a bridge over a canal is in the form of a circular arc of radius 50 ft. What is the greatest speed (in miles per hour) at which a motor-cycle can cross the bridge without leaving the ground at the highest point ?

9. A particle suspended vertically from a fixed point by a light string 4 ft. long is projected horizontally with such a velocity that the string slackens when the particle is 6 ft. above its lowest point. Find how much higher it will rise.

10. A heavy particle is allowed to slide down a smooth vertical circle of radius $27a$ from rest at the highest point. Show that on leaving the circle it moves in a parabola whose latus rectum is $16a$.

11. A particle is projected along the inside of a smooth vertical circular hoop from its lowest point with such a velocity that it leaves the hoop and returns to the point of projection again. Find the velocity of projection and determine where the particle leaves the hoop, if a be the radius of the hoop.

12. Two particles, m_1 and m_2 being simultaneously to slide down a smooth circular tube whose plane is vertical, starting from the extremities of a horizontal diameter, so that

they collide at the lowest point. If h_1, h_2 are the vertical heights to which they rise after impact, show that

$$\frac{h_1}{h_2} = \frac{\{(2e+1)m_2 - m_1\}^2}{\{(2e+1)m_1 - m_2\}^2},$$

where e is the coefficient of restitution between the masses.

ANSWERS

1. 2 lbs. wt. 3. (i) 5 lbs. wt. (ii) 20 lbs wt. 8. $27\frac{3}{11}$.
9. 9 inches. 11. $\sqrt{\frac{1}{2}ga}$; at an angular distance of 120° from the lowest point,

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CHAPTER XVI

MOTION ON A ROUGH PLANE

16·1. When a body in contact with a rough surface has a sliding motion on the surface, it experiences a resisting force at its point of contact tending to oppose its motion which is known as the force of friction.

The laws which this force of dynamical friction satisfies are

(i) *the direction of friction is, at every instant, just opposite to the direction in which the point of contact slides on the surface* ;

(ii) *the magnitude of the force of dynamical friction bears a constant ratio to the normal reaction of the surface on the body at the point of contact, this constant ratio being known as the* **coefficient of friction** (dynamical). Thus, if μ be the coefficient of friction, and R the normal reaction on the body, the force of friction F at the point of contact is given by

$$F=\mu R.$$

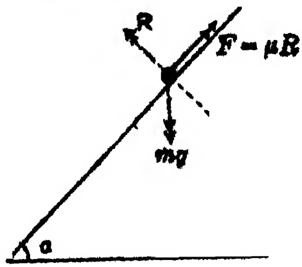
The angle λ, such that $\tan\lambda=\mu$, is known as the *angle of friction.*

Note. When the point of contact does not slip on the rough surface, the force of friction is just sufficient to prevent the slipping motion, and is always less than μR.

16·2. ***A heavy particle sliding on a rough inclined plane.***

When a particle of mass m slides down a rough inclined plane of inclination α to the horizon, along the line of

greatest slope, if R be the normal reaction of the plane and μ be the coefficient of friction between the particle and the plane, then the force of friction $F = \mu R$ is upwards. Also, the weight mg of the particle (whose mass is assumed to be m) acts vertically downwards.



Hence, since the particle has no motion perpendicular to the plane, the resultant force perpendicular to the plane is zero, $\therefore \quad R = mg \cos a$.

Now, the resultant force down the plane

$$= mg \sin a - F = mg \sin a - \mu R$$

$$= mg (\sin a - \mu \cos a).$$

Hence, *the acceleration of the particle down the plane is*

$$\mathbf{g (\sin \alpha - \mu \cos \alpha).}$$

When the particle moves upwards on the plane along the line of greatest slope, for instance, when it is projected upwards with any velocity, the force of friction, which is, as before, $\mu mg \cos a$ in magnitude, acts down the plane. Thus, the resultant force down the plane is now

$$mg \sin a + \mu mg \cos a = mg (\sin a + \mu \cos a).$$

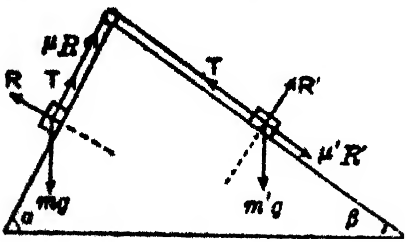
Hence, in this case, *the acceleration of the particle down the plane is*

$$\mathbf{g (\sin \alpha + \mu \cos \alpha).}$$

We thus see *that for a particle moving on a rough* inclined plane, the accelerations in the two cases when it rises up, and when it moves downward, are different. Thus, the times of ascent and *descent of* a particle projected up a rough inclined plane will be different ; also, the velocity of the particle on coming back to the point of projection will be different from the starting velocity. Moreover, in this case, the sum-total of the kinetic and potential energies will not remain constant.

16·3. *Two rough inclined planes of inclinations α and β to the horizon and of equal height are placed back to back. Two particles of masses m and m' are placed one on each plane, and are connected by a light inextensible string passing over a small smooth pulley placed at the common vertex of the planes; μ and μ' being the respective coefficients of friction of the two planes, to determine the common acceleration of the system, assuming that m, placed on the first plane, descends.*

Let R and R' denote the normal reactions of the planes, T the tension of the string, and f denote the common acceleration of the system.



As there is no motion of m perpendicular to the plane,

$$R = mg \cos \alpha. \quad \cdots \text{(i)}$$

Since m descends, the frictional force μR acts upwards on it.

Now, considering the total force on m down the plane,

$$mg \sin \alpha - T - \mu R = mf,$$

or, using (i),

$$mg (\sin \alpha - \mu \cos \alpha) - T = mf. \quad \cdots \text{(ii)}$$

In a similar manner noting that m' ascends on the second plane, and so the frictional force on m' is $\mu' R'$ downwards, we ultimately get

$$T - m'g (\sin \beta + \mu' \cos \beta) = m'f. \quad \cdots \text{(iii)}$$

From (ii) and (iii), adding,

$$(m + m') f = g \{m (\sin \alpha - \mu \cos \alpha) - m' (\sin \beta + \mu' \cos \beta)\}$$

giving the required acceleration of the system.

Note. In order that f may be positive, in other words, m may actually have a downward motion, we must have,

$$m(\sin a-\mu\cos a) > m'(\sin\beta+\mu'\cos\beta)$$

i.e., $$\frac{m}{m'} > \frac{\sin\beta+\mu'\cos\beta}{\sin a-\mu\cos a}.$$

In a similar way, m' will descend, or m will ascend, if

$$\frac{m'}{m} > \frac{\sin a+\mu\cos a}{\sin\beta-\mu'\cos\beta},$$

or, $$\frac{m}{m'} < \frac{\sin\beta-\mu\cos\beta}{\sin a+\mu\cos a}.$$

If $$\frac{\sin\beta-\mu'\cos\beta}{\sin a+\mu\cos a} < \frac{m}{m'} < \frac{\sin\beta+\mu\cos\beta}{\sin a-\mu\cos a},$$

there will be no motion of the system.

Examples on Chapter XVI

1. A ball is projected along a rough horizontal plane with a velocity of 16 ft. per sec. If the coefficient of friction be $\frac{1}{4}$, find how far the ball will go before coming to rest.

2. A mass of 8 lbs. hanging freely over the edge of a rough horizontal table draws by means of a string a mass of 4 lbs. along the table through a distance of 20 ft. in $1\frac{1}{2}$ secs. Find the coefficient of friction of the table.

3. A body of mass 20 lbs. is sliding down a rough inclined plane whose coefficient of friction is $\frac{1}{4}$ and whose elevation is $\sin^{-1}\frac{3}{5}$ with a velocity of 16 ft. per sec. What force will stop it in 80 feet ?

✓4. A particle slides down a rough inclined plane whose elevation is 45° and coefficient of friction $\frac{3}{4}$. Show that the time it takes to travel any distance down the plane is twice what it would have taken if the plane were smooth.

5. A ball is projected with a velocity of 64 ft. per sec. up a rough plane of inclination 60° and angle of friction 30°. Find the velocity and the time when it reaches the point of projection again.

6. Two rough planes of elevation 30° and 60° and of the same height are placed back to back. A mass of 8 lbs. is placed on the first plane and that of 24 lbs. on the second plane, and the two are connected by a light string passing over a smooth pulley at the top of the planes. If the coefficient of friction is $1/\sqrt{3}$ for either plane, find the resulting acceleration.

7. A rough plane is 100 feet long and 60 feet high, the coefficient of friction being $\frac{1}{2}$. If a particle projected up the plane from the bottom just reaches the top, find its initial velocity.

8. A heavy slab of uniform thickness of mass M, whose under-surface is rough but the upper surface smooth, slides down a given inclined plane of elevation a. Find the acceleration with which a particle of mass m laid on its upper surface will move along the slab, if μ be the coefficient of friction.

9. Two bodies of masses 10 lbs. and 5 lbs. are connected by a light inextensible string; the first is placed on a rough horizontal table of coefficient of friction $\frac{3}{10}$, and the string after passing over a light smooth pulley at the edge of the table supports the second body, which hangs vertically. Find the acceleration of the bodies and the tension of the string.

10. A ball is projected up a rough inclined plane of elevation $\frac{1}{4}\pi$, with velocity u, and returns to its starting point with velocity v. If t_1, t_2 are the times of ascent and descent, and μ the coefficient of the friction, show that

$$\mu = \frac{u^2 - v^2}{u^2 + v^2} = \frac{t_2^2 - t_1^2}{t_2^2 + t_1^2}.$$

11. A particle is projected with velocity u up a rough plane of elevation a, which passes through the point of projection. If the angle of friction λ be $< a$, and if the particle reaches the point of projection again with velocity v, then

$$v = u\sqrt{\left\{\frac{\sin(a-\lambda)}{\sin(a+\lambda)}\right\}}.$$

12. A ball is thrown with velocity u up a rough plane of elevation θ. If $\lambda\ (< \theta)$ be the angle of friction, show that the ball again has the velocity u, when it is at a distance

$$\frac{u^2}{g} \cdot \frac{\sin 2\lambda \cos \theta}{\cos 2\lambda - \cos 2\theta}$$

from the point of projection.

13. Two particles are projected with equal velocities, one straight up and the other straight down a rough plane of elevation α and angle of friction $\lambda\ (> \alpha)$. If s_1 and s_2 are the distances travelled by the two bodies, then

$$\frac{s_1}{s_2} = \frac{\sin (\lambda - \alpha)}{\sin (\lambda + \alpha)}.$$

14. If PL be the vertical chord through any point P on a vertical circle and if the times of sliding down all those chords of the circle through P which are on the side of PL remote from the centre be equal, then show that the chords are equally rough.

15. A body of 20 lbs. wt. slides from rest through a distance of 100 ft. down a rough plane whose elevation is $\tan^{-1} \frac{7}{24}$ and coefficient of friction $\frac{1}{4}$. Find the work done on the mass by the forces acting on it.

16. A car takes a banked corner of a racing track at a speed v, the lateral gradient α being designed to reduce the tendency to side-slip to zero for a lower speed u. Show that the coefficient of friction necessary to prevent side-slip for the greater speed v must be at least

$$\frac{(v^2 - u^2) \sin \alpha \cos \alpha}{v^2 \sin^2 \alpha + u^2 \cos^2 \alpha}.$$

ANSWERS

1. 16 ft. 2. $\frac{1}{2}$. 3. 9 lbs. wt.

5. $32\sqrt{2}$ ft./sec., $\sqrt{3}(1+\sqrt{2})$ sec. 6. $8(\sqrt{3}-1)$ ft./sec^2.

7. 80 ft./sec. 8. $\mu\left(1+\frac{m}{M}\right) g \cos \alpha$.

9. 8·84 ft./sec^2 ; 140·8 poundals. 15. 80 ft.-lbs.

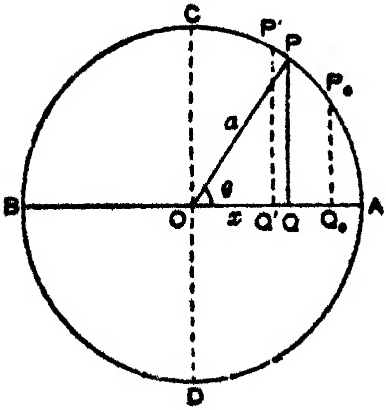
CHAPTER XVII

SIMPLE HARMONIC MOTION AND SIMPLE PENDULUM

171. Simple Harmonic Motion (or S. H. M.)

If a point moves uniformly in a circle and if a second point moves in a fixed diameter of that circle so as always to be at the foot of the perpendicular from the first point on the diameter, then the motion of the second point is known as a simple harmonic motion.

Let a point P move with a uniform angular velocity ω in a circle of radius a with centre O, and let the point Q be always at the foot of the perpendicular PQ on a fixed diameter AOB. Then, as P starts from A, Q also starts from A. When P moves along ACB and comes to B, Q moves along AOB and reaches B. As P continues its motion along BDA, Q turns back and traces the path BOA, reaching A with P. The motion of Q along AOB is thus oscillatory. This motion is defined as a simple harmonic motion.



O is clearly the *centre of oscillation* and the maximum distance OA or $OB = a$ to which Q moves on either side of O is called the *amplitude*.

As P evidently takes the time $\frac{2\pi}{\omega}$ to complete the

circle, Q also takes the same time to complete one oscillation, *i.e.*, to move from one extreme position to the other and back. This interval $\frac{2\pi}{\omega}$ is called the *periodic time*.

Velocity and acceleration of Q.

At any instant t, let x be the distance of Q from O, and angle $POQ = \theta$ say.

After an infinitely short time, P going to P', Q comes to Q', and it is evident that the displacements of P and Q parallel to AB are equal, since PQ is always perpendicular to AB.

Thus, the velocity (rate of displacement) of Q, along AB at any instant is exactly equal to the component of the velocity of P parallel to AB.

But the velocity of P is ωa along the circumference PP', *i.e.*, perpendicular to OP, and its component parallel to AB is $\omega a \cos(90° - \theta) = \omega a \sin\theta = \omega.PQ = \omega\sqrt{a^2 - x^2}$.

Hence, *the velocity of* Q *at a distance* x *from* O $= \omega\sqrt{a^2 - x^2}$.

Velocity at the distance a is thus zero.

Again, since at every instant the component velocity of P along AB is equal to that of Q, the acceleration (rate of change of veloc.) of Q along AB is equal to the component of acceleration of P parallel to AB. But the acceleration of P at any instant is $\omega^2 a$ along PO, of which the component parallel to AB is $\omega^2 a \cos\theta = \omega^2 x$.

Hence, *the acceleration of* Q *at a distance* x *from* O $= \omega^2 x$ directed towards O.

This leads to a formal definition of simple harmonic motion as follows :

If a point moves along a straight line AOB in such a manner that its acceleration is always directed towards

a fixed point O on it and is at any instant proportional to its distance from O, then the motion of the point is defined to be a simple harmonic motion.

If at a distance x from O the acceleration of a point Q moving along a straight line AOB be μx towards O, then comparing with the above result, we may imagine an auxiliary point P to move in a circle with uniform angular velocity $\omega = \sqrt{\mu}$, such that Q will always remain at the foot of the perpendicular from P on the line. Hence, at a distance x from O,

veloc. of $Q = v = \sqrt{\mu(a^2 - x^2)}$, when a is the amplitude, *i.e., the extreme distance from O from which Q starts from rest.*

Period $T = \dfrac{2\pi}{\sqrt{\mu}}$, and is independent of the amplitude.

If time be measured from the instant when the particle is at its extreme position A, then at any instant t, $a = \angle AOP = \sqrt{\mu}t$, and the position of Q is given by

$$x = a \cos \sqrt{\mu}t.$$

If on the other hand the time be measured from any instant, for instance when Q is at Q_0 or P is at P_0 where $\angle AOP_0 = \epsilon$ then at any time t, $\angle P_0OP = \sqrt{\mu}t$, and so, $\angle AOP = \sqrt{\mu}t + \epsilon$.

Then, the position of Q is given by

$$x = a \cos (\sqrt{\mu}t + \epsilon).$$

The angle ϵ is called the *epoch*, and the time $t + \dfrac{\epsilon}{\sqrt{\mu}}$ from the extreme position A at any instant (usually expressed as a fraction of the periodic time) is defined as the *phase* at that instant.

As two important illustrations of simple harmonic motion we shall discuss :

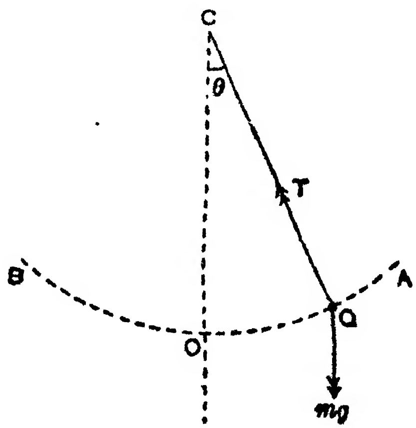
(i) The motion of a simple pendulum oscillating through a small angle.

(ii) Oscillatory motion of a particle attached at the extremity of an elastic string (or light spring) stretched along its length.

17·2. Simple Pendulum.

A heavy particle hanging from a fixed point by a light inextensible string, and made to oscillate in a vertical plane, is called a simple pendulum.

Let Q be a particle of mass m hanging from the point C by the string CQ of length l. If the string be drawn



aside to the position CA at a small angle (say α) to the vertical CO, and then let go, the particle Q will describe an arc of a vertical circle with centre C. Now, if θ be the angle QCO at any instant, the forces acting on the particle are the tension T along the string QC, and the weight mg vertically downwards. As there is no motion of Q along CQ, the forces along this direction balance. The only force left on Q is the component $mg \sin \theta$ along the tangent at Q to the arc QO.

Thus, the acceleration of Q is $g \sin \theta$ along the tangent to the arc QO towards O.

Now, if a, and so θ, be small, we may take $\sin\theta = \theta = \frac{x}{l}$, where x is the length of the arc OQ. Also in this case the small arc $AQOB$ may be taken to be practically a straight line, on which the motion of Q is with an acceleration towards a fixed point O, of magnitude $g\frac{x}{l}$, at a distance x from O, that is proportional to the distance from O.

The motion of the particle Q is therefore a simple harmonic oscillation about O and the periodic time or the time of oscillation of the pendulum, is given by

$$T = 2\pi\Big/\sqrt{\frac{g}{l}} = 2\pi\sqrt{\frac{l}{g}}.$$

where, the length of the string CQ, is known as the *length of the pendulum.*

Note 1. The time of oscillation of a pendulum oscillating at a small angle with the vertical, depends, as seen above, on the length of the pendulum, but is independent of the amplitude of oscillation; in other words, *it does not depend on the angle from which the pendulum starts to swing, provided this angle is small.*

Note 2. A pendulum which swings from one extreme position of rest to the other in one second, that is which makes a complete oscillation in two seconds, is called a **Seconds Pendulum.**

For such a pendulum, the length is given by $2 = 2\pi\sqrt{\frac{l}{g}}$ or, $l = g/\pi^2$, and taking the value of $g = 32$ ft./sec^2 or 981 cms/sec^2, we get approximately: $l = 39$ inches or 99·4 cms.

A swing from one extreme position to the other, *i.e.*, half a complete oscillation is called a *beat*. A seconds pendulum then beats seconds.

Note 3. The above formula for T enables us to compare the values of 'g' at two places on the earth by observing the periods of oscillation of a simple pendulum of given length l at the two places, *for*, $g_1/g_2 = T_2^2 : T_1^2$ in this case.

175. Determination of heights, or depths from the earth's surface, by simple pendulum.

Newton's Universal Law of Gravitation states shat

"*Even particle of matter in this universe attracts every other particle with a force which is proportional to the product of their masses, and varies inversely as the square of the distance between them*".

Assuming the earth to be approximately a homogeneous solid sphere, the consequence of the above law is (as has been shown in any treatise on the theory of attractions) that (*i*) *the resultant attraction of the earth per unit mass at any external point is inversely proportional to the square of its distance from the centre*, and (*ii*) *at any internal point the attraction is directly proportional to the distance from the centre.*

Thus, if g be the value of acceleration due to gravity on the earth's surface (at sea-level), g_1 its value at a height h above the surface of the earth (supposed spherical, of radius a), then

$$\frac{g_1}{g}=\frac{1}{(a+h)^2}:\frac{1}{a^2}=\frac{a^2}{(a+h)^2}.$$

Again, if g_2 be the value at a depth d below the earth's surface (*e.g.*, at the bottom of a mine),

$$\frac{g_2}{g}=\frac{a-d}{a}.$$

Hence, if the periods of oscillation of a simple pendulum of a given length l at sea-level (on the earth's surface), at a height h above the surface of the earth, and at a depth d below the surface of the earth, be respectively T, T_1 and T_2, we get

$$T_1:T=2\pi\sqrt{\frac{l}{g_1}}:2\pi\sqrt{\frac{l}{g}}=\sqrt{\frac{g}{g_1}}=\frac{a+h}{a}$$

and similarly,

$$T_2 : T = \sqrt{\frac{g}{g_2}} = \sqrt{\frac{a}{a-d}}.$$

Thus, $$h = a\left(\frac{T_1}{T} - 1\right),$$

and $$d = a\left(1 - \frac{T^2}{T_2{}^2}\right).$$

17·4. Oscillations of a particle attached to an elastic string (or spiral spring).

When an elastic string is stretched (for instance by keeping one extremity fixed and pulling at the other, or hanging it vertically at one extremity, and suspending a heavy particle from the other), the tension in the string is given from an experimental law known as **Hooke's law** which states that :

The tension in a stretched elastic string is proprtional to its extension per unit length.

Mathematically, if l be the natural (unstretched) length of the string, $l+x$ its extended length, T being the tension in the string in this case,

$$T = \lambda \frac{x}{l},$$

where λ is called the *modulus of elasticity* of the string.

The extension or compression of a spiral spring follows the same law, but in this case by its length we mean the length of the axis of the spiral, and not the actual length of the wire which is twisted to form the spring. Moreover, in this case, when the spring is compressed, the extension is negative, so that tension is negative, in other words, it is a thrust whose measure is given by the same law.

Now, if a particle of mass m be tied at the extremity of an elastic string (or a spiral spring) the other extremity of which is fixed, and the string be extended and then let go (the whole system lying on a smooth horizontal table), the particle will oscillate, performing a S.H.M. ; for in this

case the forces of the particle being the tension of the spring, its value in any position P of the particle is $\frac{\lambda}{ml}x$, where x = the total increment in length = OP, O denoting the position of the particle when the string is just unstretched (and is thus a fixed point). The acceleration of the particle being here $\frac{\lambda}{ml}x$, and directed towards O, the motion is a S.H.M., the period of oscillation being $2\pi\sqrt{\frac{ml}{\lambda}}$. [See § *17·1*]

17·5. Illustrative Examples.

Ex. 1. *A particle of mass 4 lbs. executing simple harmonic oscillation, has velocities 8 ft./sec. and 6 ft./sec. respectively, when it is at distances 3 ft. and 4 ft. from the centre of its path. Find its period and amplitude.*

Find also the force acting on the particle when it is at a distance of 1 foot from the centre.

As the particle executes S.H.M., assume μx to be its acceleration at a distance x from the centre of oscillation. Then, a being its amplitude, its velocity at the distance x from the centre is known to be given by

$$v = \sqrt{\mu(a^2 - x^2)}.$$

Thus, from the given data,

$$8 = \sqrt{\mu(a^2 - 3^2)} \qquad \cdots \qquad \cdots \qquad \text{(i)}$$

and

$$6 = \sqrt{\mu(a^2 - 4^2)}, \qquad \cdots \qquad \cdots \qquad \text{(ii)}$$

From these, by division,

$$\frac{4}{3} = \sqrt{\frac{a^2 - 9}{a^2 - 16}},$$

whence ultimately, $a^2 = 25$, or $a = 5$ ft. giving the amplitude.

Hence, from (i), $8 = 4\sqrt{\mu}$, or, $\mu = 4$.

Now, the period,

$$T = \frac{2\pi}{\sqrt{\mu}} = \frac{2\pi}{2} = \pi \text{ seconds.}$$

Again, the acceleration at distance 1 foot from the centre is $\mu . 1 = 4$ ft./sec^2.

Therefore, the force here acting on the particle of mass 4 lbs. is $4 \times 4 = 16$ poundals.

Ex. 2. *The length of the pendulum of a clock is 39 inches, and the clock gains 10 seconds in a day at a place on earth, by how much must the length of the pendulum be altered in order to correct the clock?*

As seconds in a clock are indicated by beats of its pendulum, in one day *i.e.*, $24 \times 60 \times 60 = 86400$ seconds, the pendulum of a correct clock should make as many beats. For our given clock, which gains 10 seconds a day, the number of beats in a day is $86400 + 10 = 86410$, so that the period of a complete oscillation is

$$2 \times \frac{86400}{86410} \text{ seconds.}$$

Thus, $$2 \times \frac{86400}{86410} = 2\pi \sqrt{\frac{l}{g}}, \text{ where } l = 39 \text{ inches.} \quad \cdots \quad \text{(i)}$$

Also for a correct clock, the length l' of the pendulum is given by

$$2 = 2\pi \sqrt{\frac{l'}{g}}. \quad \cdots \quad \cdots \quad \text{(ii)}$$

$$\therefore \quad \sqrt{\frac{l'}{l}} = \frac{8641}{8640} = 1 + \frac{1}{8640},$$

or, $$\frac{l'}{l} = \left(1 + \frac{1}{8640}\right)^2 = 1 + \frac{2}{8640} \text{ approximately,}$$

giving, $$l' - l = \frac{l}{4320} = \frac{39}{4320} = \cdot 009 \text{ inches approximately.}$$

Thus, the length of the pendulum should be increased by ·009 inches.

Ex. 3. *A clock which keeps correct time on the surface of the earth loses 20 seconds a day when taken to the top of a hill. Find the height of the hill, assuming the radius of the earth to be 4000 miles.*

Let h miles be the height of the hill, and g and g' the accelerations due to gravity on the surface of the earth and at the top of the hill respectively.

Then, (as in Art. 17·3),

$$\frac{g}{g'}=\left(\frac{4000+h}{4000}\right)^2. \quad \dots \quad \dots \quad \text{(i)}$$

Now, for the correct clock, the time of a complete oscillation being 2 secs.,

$$2=2\pi\sqrt{\frac{l}{g}}. \quad \dots \quad \dots \quad \text{(ii)}$$

Also at the top of the hill, in one day *i.e.*, 86400 seconds, the clock losing 20 seconds, it makes 86380 beats, and so the time of a complete oscillation is

$$2\times\frac{86400}{86380}=2\pi\sqrt{\frac{l}{g'}}. \quad \dots \quad \dots \quad \text{(iii)}$$

From (ii) and (iii), using (i),

$$\frac{8640}{8638}=\sqrt{\frac{g}{g'}}=\frac{4000+h}{4000}.$$

$$\therefore \quad h=4000\left(\frac{8640}{8638}-1\right)=4000\times\frac{2}{8638} \text{ miles}$$

$$=4890 \text{ feet (nearly).}$$

Ex. 4. *An elastic string of natural length l and modulus of elasticity λ hangs vertically from one extremity and at the other end a particle of mass m is suspended. The particle is held with the string just unstretched and then let go. Show that it performs a simple harmonic oscillation, and find the period.*

The starting position of the particle being O, when the string is just unstretched, let x be the depth of the particle below O at any moment. Then the tension of the string by Hooke's Law, is clearly $\lambda x/l$. Hence, the resultant force acting on the particle vertically upwards is

$$\lambda\frac{x}{l}-mg=\frac{\lambda}{l}\left(x-\frac{mgl}{\lambda}\right).$$

If we take a point O' at a depth mgl/λ below O, then O' is also a fixed point, and x' denoting the depth of the particle below O', the force on the particle in this position is $\frac{\lambda}{l}\cdot x'$.

Hence, the acceleration of the particle at a distance x' from O is $\frac{\lambda}{ml} \cdot x'$ towards O' i.e., proportional to This identifies the motion of the particle to be S.H.M with O' as centre, and the period of oscillation

$$T = 2\pi \Big/ \sqrt{\frac{\lambda}{ml}} = 2\pi \sqrt{\frac{ml}{\lambda}} \quad [\textit{ Cf. Art. 174 }]$$

Examples on Chapter XVII(a)

1. A particle moving with S.H.M. has a velocity of 8 ft. per sec. when at a distance of 3 ft. from the centre of its path, and has a velocity of 6 ft. per sec. when at a distance 4 ft. Find its maximum velocity and periodic time.

2. A particle moving with S.H.M. has a maximum velocity v. Find the velocity of the particle (i) when it is half way between the centre and the extreme position and (ii) also when half the time has elapsed from the centre to the extreme position

3. A particle performing harmonic oscillations in a straight line starts at a point 14 ft. from the centre of its path, and has a maximum velocity of 22 ft. per sec. ; find its periodic time.

4. A horizontal shelf moves vertically with S.H.M. of period 2 secs. Find the greatest amplitude in centimetres that it can have so that books resting on it may always be in contact with it.

5. A particle moving with S.H.M. in a straight line has velocities v_1, v_2 at distances x_1, x_2 from the centre of its path. Show that if T be the period of its motion,

$$T = 2\pi \sqrt{\frac{x_1^2 - x_2^2}{v_2^2 - v_1^2}}. \qquad [\textit{ C. P. 1964, '69 }]$$

6. A particle oscillating harmonically in a straight line has velocities v_1, v_2 and accelerations f_1, f_2 in two of its

positions on the path. If d be the distance between the two positions, show that

$$d=\frac{v_1^2-v_2^2}{f_1+f_2}.$$

7. A particle is executing S.H.M. between two points A and B. If the period of oscillation be 2π, and if v be the velocity of the particle at any point P on its path, then show that

$$v^2=AP.BP.$$

8. A seconds pendulum gains 18 secs. a day at sea-level. To what height it must be elevated in order to keep true time?

9. A clock which gains 13 seconds a day at a place on the surface of the earth loses 10 seconds a day when taken down at the bottom of a mine. Compare the force of gravity at these two places.

10. If a seconds pendulum be lengthened by $\frac{1}{100}$th of its length, how many seconds will it lose in a day?

11. A pendulum, when carried to the top of a mountain, is observed to lose in a given time just twice as much as it does, when taken to the bottom of a mine in the neighbourhood. Show that the height of the mountain is equal to the depth of the mine.

12. At the end of three successive seconds, the distances of a point moving with S.H.M. from its mean position, measured in the same direction, are 1, 5 and 5. Show that the period of a complete oscillation is

$$\frac{2\pi}{\cos^{-1}\frac{3}{5}}.$$

13. A body performing S.H.M. in a straight line OPQ has its velocity zero when at points P and Q whose distances from O are x and y respectively, and has velocity v when half-way between them. Show that the complete period is

$$\frac{\pi(y-x)}{v}.$$

14. A particle is executing a Simple Harmonic Motion in a straight line. The periodic time of the particle is T. The particle takes time T_1 to move from the position of the maximum displacement to one in which the displacement is half the amplitude. Show that

$$6T_1 = T.$$

15. In a S.H.M., if f be the acceleration and v the velocity at any time, and T is the periodic time, then

$$f^2T^2 + 4\pi^2v^2$$

is constant.

16. A particle is performing a S.H.M. of period T about a centre O, and it passes through a point P with a velocity v in the direction OP. If OP be equal to x, and if the particle returns to P in time t, then

$$t = \frac{T}{\pi}\tan^{-1}\frac{vT}{2\pi x}.$$

17. A spiral spring 2 ft. long is hung up at one end. Its length would be doubled by a steady pull of 6 lbs. wt. A wt. of 3 lbs. is hung to the lower end and let go. Find how far it falls before first coming to rest, and the time of a complete oscillation.

18. Two unequal weights are hanging together at one end of an elastic string, and one of them falls off. Show that the other will perform simple harmonic oscillations or not according as the one which falls off is the lighter or the heavier of the two.

19. If a body of mass m executing S.H.M. makes n complete oscillations per sec., show that the difference of its K.E. when at the centre, and when at a distance x from the centre, is given by

$$2m\pi^2n^2x^2.$$

20. A smooth airless tunnel is bored through a diameter of the earth. Find the time taken by a particle to slide through it, and the speed at the centre.

21. A flat plate oscillates vertically through a distance of 4 inches. Find the greatest number of vibrations per minute so that a particle resting on it is not jerked off.

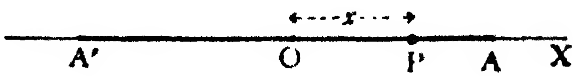
22. A particle executes S.H.M. along the line AB. If C divides AB in the ratio 3 : 1, show that the particle takes twice as long to describe AC as to describe CB.

ANSWERS

1. 10 ft. per sec.; π secs. 2. $\frac{1}{2}\sqrt{3}v$, $\frac{1}{2}\sqrt{2}v$. 3. 4 secs.
4. 99·8 cm. 8. 4400 ft. 9. 8648 : 8689. 10. 482.
17. 2 ft.; $\frac{\sqrt{2\pi}}{4}$. 20. 42 min.; 26000 ft./sec.
21. $30\sqrt{6g}/\pi$.

17·6. Simple Harmonic Motion (Analytical Treatment).

A particle starts from rest at a distance a from a fixed point O on a straight line, and moves with an acceleration which is always directed towards O and varies as the distance from O. Investigate the motion, and prove that the motion is oscillatory about O.



Let a particle starting from rest at A (where $OA = a$) move with an acceleration directed towards O, and be at P at a distance x from O at any instant t. The acceleration here is μx towards O, where μ is a positive const. i.e., $-\mu x$ along OX (in the direction of x increasing). Therefore, the equation is then

$$\frac{d^2x}{dt^2} = -\mu x \, (\mu > 0). \qquad \cdots \text{(i)}$$

Multiplying both sides by $2\dfrac{dx}{dt}$ and integrating with respect to t, we get

$$\left(\frac{dx}{dt}\right)^2 = -\mu x^2 + C \qquad \text{(ii)}$$

where the integration constant C is found from the initial condition that $\dfrac{dx}{dt} = 0$ when $x = a$, whence

$$0 = -\mu a^2 + C \text{ or } C = \mu a^2.$$

$\therefore$ from (ii), at any instant,

$$\left(\frac{dx}{dt}\right)^2 = \mu(a^2 - x^2). \qquad \text{(iii)}$$

$$\therefore \quad \frac{dx}{dt} = -\sqrt{\mu(a^2 - x^2)} \qquad \text{(iv)}$$

the negative sign being taken here, since the particle moves towards O whereas $\dfrac{dx}{dt}$ represents the velocity along OX (in the direction of x increasing), and is accordingly negative here.

As the particle moves towards O, x diminishes, and so from (iv), the velocity gradually increases in magnitude. At O, where $x = 0$, the velocity is $a\sqrt{\mu}$ along AO, and the particle passes towards the negative side of O. But here the acceleration being directed towards O, opposes the motion and the velocity gradually diminishes.

Here at any point, x is negative, and thus the positive value of the acceleration is $\mu(-x) = -\mu x$ towards O, *i.e.*, in the direction of x increasing. Hence, the equation of motion here is $\dfrac{d^2x}{dt^2} = -\mu x$, which is the same as (i).

Thus, whether the particle is to the right or left of O, we get the equation of motion continuing namely (i), and accordingly, on integration, we get (ii). As $\dfrac{dx}{dt} = -a\sqrt{\mu}$

when $x=0$, we get the same $C=\mu a^2$ on the left of O, as on the right, and thus (iii) holds on the left of O as well.

To the left of O, when $x=-a$, $\frac{dx}{dt}=0$, so that the particle comes to rest at A' where $A'O=a=OA$.

Starting from rest at A', with the acceleration directed towards O, exactly the same motion is repeated in a reverse direction, and the particle again comes to rest at A. The motion is repeated over and over again. Hence, the motion is oscillatory about O. This motion is known as *simple harmonic oscillation.*

The greatest distance from O to which the particle proceeds is a on either side, and this is called the *amplitude* of the oscillation.

The total time of one complete oscillation from A to A' and back to A, is called the *period* of oscillation.

To find this period, we note that since the motion is symmetrical about O, time from A to O = that from O to A', and time from A' to A again is exactly equal to that from A to A'. Hence, the period of one complete oscillation is 4 times the time from A to O.

Now, from (iv), $-\frac{dx}{\sqrt{a^2-x^2}}=\sqrt{\mu}\,dt.$

$\therefore$ integrating, $\cos^{-1}\frac{x}{a}=\sqrt{\mu}t+D$

where, $\because$ when $x=a$, $t=0$, we get $D=0$.

$$\therefore \quad \cos^{-1}\frac{x}{a}=\sqrt{\mu}t, \quad \text{or, } x=a\cos\sqrt{\mu}t. \qquad \cdots \quad \text{(v)}$$

When $x=0$, *i.e.*, at O, $\cos\sqrt{\mu}t=0$,

or, $\sqrt{\mu}t=\frac{\pi}{2}$ *i.e.*, $t=\frac{\pi}{2\sqrt{\mu}}$.

Hence, the period of oscillation $=4\cdot\frac{\pi}{2\sqrt{\mu}}=\frac{2\pi}{\sqrt{\mu}}$.

Equation (iv) gives the velocity at any distance, and equation (v) gives the position at any time. Differentiating (v) we get velocity corresponding to any time.

When a particle moves on a straight line with an acceleration always directed towards a fixed point O on it and is proportional to the distance from O, the motion will be one of oscillation about O whatever the initial conditions may be, and this motion is known as a **simple harmonic motion**, or briefly, S.H.M. (*Compare* § *17·1*)

17·7. A general solution of a simple harmonic oscillation under any initial conditions.

Let x denote the distance measured from O, giving the position of the particle at any instant t, measured from any suitable moment.

The differential equation for a simple harmonic motion is, according to the definition,

$$\frac{d^2x}{dt^2} = -\mu x, \qquad \dots \qquad \dots \quad \text{(i)}$$

where μ is a positive constant.

[This equation of motion holds on the right as well as on the left of O, as explained in Art. 17·6.]

Multiplying both sides of (i) by $2\frac{dx}{dt}$ and integrating with respect to t,

$$\left(\frac{dx}{dt}\right)^2 = -\mu x^2 + C, \qquad \dots \quad \text{(ii)}$$

where C is an arbitrary integration constant, which for actual motion must clearly be positive (for otherwise square of the velocity being negative for any position, velocity becomes imaginary). Assuming therefore, $C = \mu A^2$, where A is an arbitrary constant, equation (ii) then becomes

$$\left(\frac{dx}{dt}\right)^2 = \mu(A^2 - x^2). \qquad \dots \qquad \dots \quad \text{(iii)}$$

$$\therefore \quad \frac{dx}{dt} = \pm \sqrt{\mu}\sqrt{A^2 - x^2}.$$

$$\therefore \quad \mp \frac{dx}{\sqrt{A^2 - x^2}} = \sqrt{\mu}\, dt.$$

Integrating (putting $x = A \cos \theta$ for integrating the left side),

$$\mp \cos^{-1} \frac{x}{A} = \mu t + \varepsilon,$$

where ε is an arbitrary constant of integration.

Hence, $\frac{x}{A} = \cos \{\pm (\sqrt{\mu} t + \varepsilon)\}$

or, $\quad \mathbf{x = A \cos (\sqrt{\mu} t + \varepsilon)}. \quad \cdots \quad \cdots \quad$ (iv)

This is the most general solution of the differential equation (i) for simple harmonic motion. [See *note* below]

Differentiating (iv), we get

$$\frac{dx}{dt} = -A \sqrt{\mu} \sin (\mu t + \varepsilon). \quad \cdots \quad \text{(v)}$$

In any particular case, the initial position and velocity being given, the values of x and $\frac{dx}{dt}$ are known when $t = 0$, and substituting these in (iv) and (v) we can definitely determine the unknown constants A and ε, and hence everything about the position and motion will be completely known from equations (iii), (iv) and (v).

It is apparent from (iv) that as t changes, since $\cos (\sqrt{\mu} t + \varepsilon)$ changes periodically between ± 1, x varies periodically between $\pm A$. Hence, the motion is oscillatory about O. This oscillatory motion is simple harmonic motion.

The greatest distance A to which the particle moves on either side of the centre of force O is known as the **amplitude** of the oscillation.

Again, as t increases by $\frac{2\pi}{\sqrt{\mu}}$ (or its integral multiple), $\sqrt{\mu}t+\epsilon$ increases by 2π (or its multiple), so that from (iv) and (v), x as well as $\frac{dx}{dt}$ have got their values unchanged. Hence, after every period of $\frac{2\pi}{\sqrt{\mu}}$, the particle passes through the same position with the same velocity in the same sense, so that the same position and motion are repeated over and over again at intervals of $\frac{2\pi}{\sqrt{\mu}}$. This interval $\frac{2\pi}{\sqrt{\mu}}$ is called the **periodic time** or period of oscillation.

The reciprocal of the period, *i.e.*, the number of complete oscillations per unit of time, namely $\frac{\sqrt{\mu}}{2\pi}$ is called the **Frequency**.

If initially the particle starts from rest, as in § 17·6, $\frac{dx}{dt}=0$ when $t=0$, and so from (v), $\epsilon=0$. In a general case, when $t=0$ does not correspond to the position of rest, $\epsilon \neq 0$. In such a case ϵ is called the **Epoch**, and the angle $\sqrt{\mu}t+\epsilon$ is called the **Argument**.

The instant when the particle is at its extreme position $x=A$ on the positive side, is given by $\sqrt{\mu}t+\epsilon=0$, or $t=-\frac{\epsilon}{\sqrt{\mu}}$, and this is a position of instantaneous rest where $\frac{dx}{dt}=0$. At any instant t, the interval that has elapsed since the particle was at its extreme position on the positive side, namely $t+\frac{\epsilon}{\sqrt{\mu}}$, is called the **Phase**.

To sum up, in the most general case, *for a simple harmonic motion*,

$$f \text{ or } \frac{d^2x}{dt^2}=-\mu x \text{ or } -n^2x,$$

giving the acceleration at any point ;

$$v^2 \quad \text{or} \quad \left(\frac{dx}{dt}\right)^2 = \mu(a^2 - x^2) \text{ or } n^2(a^2 - x^2),$$

giving the velocity at any point ;

$$x = a \cos(\sqrt{\mu}t + \epsilon) \text{ or } a \cos(nt + \epsilon),$$

and $\quad T = \dfrac{2\pi}{\sqrt{\mu}}$ or $\dfrac{2\pi}{n}$, giving the period.

Note. Replacing ϵ by $\epsilon - \frac{1}{2}\pi$, the *general solution* (iv) can also be written in the form $x = A \sin(\mu t + \epsilon)$, where A and ϵ are arbitrary constants.

Again, replacing $A \cos \epsilon$ by C, and $-A \sin \epsilon$ by D, where C and D are arbitrary constants, the general solution (iv) can as well be written in the form $x = C \cos \sqrt{\mu}t + D \sin \sqrt{\mu}t$.

The two arbitrary constants in any of the above two forms will be determined from the initial values of x and $\dfrac{dx}{dt}$ at time $t = 0$, as in the above article. The final form of the solution will be the same in any particular example, whichever general form of solution we start with.

17·8. Illustrative Examples.

Ex. 1. *A particle moves on a straight line under an acceleration n^2x towards a point O on the line, where x is the distance from O. Show that, if $x = a$ and $\dot{x} = u$ when $t = 0$, then at time t,*

$$x = a \cos nt + \frac{u}{n} \sin nt. \qquad \text{[C. H. 1952]}$$

Here, the equation of motion is

$$\ddot{x} = -n^2 x.$$

From § 17·7, it follows that the general solution is

$$x = A \cos(nt + \epsilon), \qquad \dots \quad \text{(i)}$$

whence $\qquad \dot{x} = -An \sin(nt + \epsilon).$

Initially, when $t = 0$, $x = a$ and $\dot{x} = u$.

Hence, $\qquad A \cos \epsilon = a \qquad \dots \quad$ (ii)

$$-An \sin \epsilon = u. \qquad \dots \quad \text{(iii)}$$

Thus, from (i), applying (ii) and (iii),

$$x = A\cos nt\cos\epsilon - A\sin nt\sin\epsilon$$

$$= a\cos nt + \frac{u}{n}\sin nt.$$

Ex. 2. *In a S.H.M., the distance of a particle from the middle point of its path at three consecutive seconds are observed to be x, y, z. Show that the time of a complete oscillation is*

$$\frac{2\pi}{\cos^{-1}\left(\frac{x+z}{2y}\right)}.$$

For a S.H.M., x being the distance from the middle point of the path (*i.e.*, the centre of oscillation) at time t, a being the amplitude, we get generally

$$x = a\cos(\sqrt{\mu}t + \epsilon). \qquad \cdots \qquad \text{(i)}$$

For the next two consecutive seconds

$$y = a\cos\{\sqrt{\mu}(t+1) + \epsilon\} \qquad \cdots \qquad \text{(ii)}$$

$$z = a\cos\{\sqrt{\mu}(t+2) + \epsilon\}. \qquad \cdots \qquad \text{(iii)}$$

From (i) and (iii),

$$x + z = a[\cos(\sqrt{\mu}t + \epsilon) + \cos\{\sqrt{\mu}(t+2) + \epsilon\}]$$

$$= a.2\cos\{\sqrt{\mu}(t+1) + \epsilon\}\cos\sqrt{\mu}$$

$$= 2y\cos\sqrt{\mu} \quad [\text{ by (ii) }].$$

$$\therefore \qquad \sqrt{\mu} = \cos^{-1}\left\{\frac{x+z}{2y}\right\}.$$

Now, the period of a complete oscillation is

$$T = \frac{2\pi}{\sqrt{\mu}} = \frac{2\pi}{\cos^{-1}\left(\frac{x+z}{2y}\right)}.$$

Examples on Chapter XVII(b)

1. A particle is projected with velocity v directly away from a fixed point at a distance a from the point of projection. If the acceleration be attractive and of magnitude $n^2 \times$ (distance from the fixed point), find the amplitude of the S.H.M.

2. A particle rests in equilibrium under the attraction of two centres of force, which attract directly as the distance, their attractions per unit of mass at unit distance

being μ and μ'. The particle is displaced towards one of them; shew that its motion is oscillatory of period $\dfrac{2\pi}{\sqrt{\mu+\mu'}}$.

3. If the distance x of a moving point P at any time t is given by
$$x = a \cos nt + \beta \sin nt,$$
where a, β and n are constants, shew that the motion of the point is simple harmonic.

4. The speed v of a moving particle along the axis of x is given by
$$v^2 = \lambda^2 (10ax - x^2 - 16a^2),$$
where a and λ are constants. Shew that the motion is simple harmonic with its centre at $x = 5a$ and amplitude $3a$. Find the time from $x = 5a$ to $x = 8a$.

5. A point L moves in a straight line with simple harmonic motion, the centre of force being at O. Let M be a point on OA (where A is one of the points where L comes to rest) such that $2OM^2 = OA^2$. Show that the time from A to M is the same as the time from M to O.

6. A particle, moving with simple harmonic motion, has got velocities u_1, u_2 and corresponding accelerations f_1, f_2 in two of its positions. Show that the distance between these to positions is $\dfrac{u_2^2 - u_1^2}{f_1 + f_2}$ and the amplitude of the motion is
$$\frac{\{(u_2^2 - u_1^2)(f_1^2 u_2^2 - f_2^2 u_1^2)\}^{\frac{1}{2}}}{f_1^2 - f_2^2}.$$

7. A particle executes S.H.M. in a straight line. If u, v, w be its velocities at three points $x = a, b, c$ and T be the periodic time, show that
$$\frac{4\pi^2}{T^2} = \frac{\begin{vmatrix} u^2 & v^2 & w^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}}{(b-c)(c-a)(a-b)}.$$

ANSWERS

1. $\sqrt{a^2 + \dfrac{r^2}{n^2}}$. 4. $\dfrac{\pi}{2\lambda}$.

17·[illegible]. Damped Harmonic Oscillations.

A particle moves in a straight line with an acceleration always directed towards a fixed point on it and proportional to its distance from it, in a medium which offers a small resistance, proportional to its velocity. To investigate the motion.

Suppose at any instant t the particle be at P at a distance x from the fixed point O, moving along OX. Its acceleration is n^2x (say) directed towards O, and the resistance of the medium to motion is k (velocity) $= k\dot{x}$ (say), where k is small.

The equation of motion of the particle is then clearly

$$\ddot{x} = -n^2x - k\dot{x}$$

or, $$\ddot{x} + k\dot{x} + n^2x = 0. \qquad \cdots \qquad \cdots \qquad \text{(i)}$$

It may be noted that the same equation of motion holds whether x is positive or negative, or whether the particle moves towards O or away from O. For example, when the particle is at P' to the negative side of O and moving away from O, the acceleration towards O, *i.e.*, in the positive direction of the x-axis, is $n^2(-x)$, and the resistance is $k(-\dot{x})$ directed in the positive direction. Hence, $\ddot{x} = n^2(-x) + k(-\dot{x})$. Similarly for all other positions and cases.

To solve the differential equation (i), as a trial solution put $x = Ce^{mt}$. We then get $m^2 + km + n^2 = 0$,

or, $$m = -\tfrac{1}{2}k \pm \tfrac{1}{2}\sqrt{k^2 - 4n^2} = -\tfrac{1}{2}k \pm i\sqrt{n^2 - \tfrac{1}{4}k^2}$$

(if k is so small compared to n such that $n^2 - \frac{1}{4}k^2$ is positive)

Thus, writing $\sqrt{n^2-\frac{1}{4}k^2}=p$, the general solution of (i) is

$$x=C_1e^{-\frac{1}{2}kt+ipt}+C_2e^{-\frac{1}{2}kt-ipt},$$

where C_1 and C_2 are arbitrary constants, and this, by adjusting constants [writing $A\cos\epsilon=C_1+C_2$ and $-A\sin\epsilon=i(C_1-C_2)$] can be put in the real form

$$x=Ae^{-\frac{1}{2}kt}\cos(pt+\epsilon), \qquad \cdots \quad \text{(ii)}$$

where A and ϵ are arbitrary constants of integration whose values are to be determined from given initial conditions.

As k is given to be small, it appears from (ii) that the motion is practically a simple harmonic motion of period $2\pi/p$ or $2\pi/\sqrt{n^2-\frac{1}{4}k^2}$ *i.e.*, very nearly equal to the natural period $2\pi/n$, but the amplitude of the oscillation, $Ae^{-\frac{1}{2}kt}$ slowly diminishes with t, and ultimately when t is very large, it tends to zero.

The effect of the small resistance of the medium on the particle, which would otherwise perform a simple harmonic oscillation of period $2\pi/n$, is to increase the period slightly, and to gradually diminish the amplitude of the oscillation, until after a long time the motion dies out.

Such a motion is termed as *damped harmonic oscillation.*

Note. When $k=2n$ (not necessarilly small), the general solution of (i) is $x=e^{-nt}(At+B)$ which represents a motion which is not oscillatory. Here also, as $t\to\infty$, $x\to 0$, and the motion ultimately dies out.

When $k>2n$, the general solution of (i) becomes

$$x=Ae^{(-\frac{1}{2}k+\frac{1}{2}\sqrt{k^2-4n^2})t}+Be^{(-\frac{1}{2}k-\frac{1}{2}\sqrt{k^2-4n^2})t}$$

and the motion is non-oscillatory. This motion also ultimately dies out as $t\to\infty$.

17 ~~10~~. Forced oscillations. Periodic disturbing force.

A particle is moving in a straight line with an acceleration n^2x towaids a fixed origin on the line and is simultaneously acted on by a periodic force $F\cos pt$ per unit mass. To investigate the motion.

The equation of motion of the particle at any instant in this case is clearly

$$\ddot{x} = -n^2x + F\cos pt$$

$$\text{or,} \qquad \ddot{x} + n^2x = F\cos pt. \qquad \cdots \quad \text{(i)}$$

The complementary function of this linear equation, *i.e.*, a general solution of the equation $\ddot{x}+n^2x=0$ is (as in § 17·7) $x = A\cos(nt+\epsilon)$.

For a particular integral of (i), assuming $x = C\cos pt$, and putting in (i), $C(-p^2+n^2)=F$. Hence, the particular integral is $x = \dfrac{F}{n^2-p^2}\cos pt \; (p \neq n)$.

Thus, the most general solution of (i) is

$$x = A\cos(nt+\epsilon) + \frac{F}{n^2-p^2}\cos pt, \qquad \cdots \quad \text{(ii)}$$

where A and ϵ are integration constants to be determined from given initial conditions, *i.e.*, from the given values of x (position) and $\dfrac{dx}{dt}$ (velocity) at time $t=0$.

It appears from (ii) that the motion of the particle here is a composition of two simple harmonic motions of periods $2\pi/n$ and $2\pi/p$ respectively.

In case where p is nearly equal to n, we have $\dfrac{F}{n^2-p^2}$ very large, whatever finite value F may have, and hence the part $\dfrac{F}{n^2-p^2}\cos pt$ due to the disturbing periodic force becomes large compared to $A\cos(nt+\epsilon)$ for the natural harmonic motion. The motion is principally therefore a periodic motion of a period equal to that of the periodic disturbing force.

Such a motion is known as a forced oscillation. When $p=n$ exactly, the complementary function of (i) remaining the same, the particular integral, *i.e.*, a particular solution of $\ddot{x}+n^2x=F\cos nt$ is not obtained as before, by assuming $x=C\cos nt$.

Here, assuming $x = Ct \sin nt$, and substituting, we get $C(-n^2 t \sin nt + 2n \cos nt) + n^2 Ct \sin nt = F \cos nt$, whence $C = \frac{F}{2n}$. Thus, $x = \frac{F}{2n} t \sin nt$ is a particular integral. Hence, the general solution of (i) here is

$$x = A \cos (nt + \epsilon) + \frac{F}{2n} t \sin nt. \qquad \cdots \quad \text{(iii)}$$

In this case the amplitude $\frac{F}{2n} t$ of the forced oscillation continually increases with t, and ultimately becomes infinitely large.

The phenomenon of the large amplitude of vibration of a body forced to vibrate with a period nearly equal to that of its free vibrations is a familiar one in the theory of sound, and is known as *resonance*.

17·11. Damped Forced oscillations.

A particle is moving in a straight line with an acceleration $n^2 x$ towards a fixed origin on the line, in a medium which offers resistance proportional to velocity (say $k\dot{x}$ *per unit mass*), *and is simultaneously acted on by a periodic disturbing force $F \cos pt$ per unit mass. To investigate the motion.*

Here the equation of motion at any instant is

$$\ddot{x} = -n^2 x - k\dot{x} + F \cos pt$$

or, $$\ddot{x} + k\dot{x} + n^2 x = F \cos pt. \qquad \cdots \quad \text{(i)}$$

The complementary function of this linear equation, *i.e.*, the general solution of $\ddot{x} + k\dot{x} + n^2 x = 0$ is, (as in § *17·9* above),

$$x = Ae^{-\frac{1}{2}kt} \cos (\lambda t + \epsilon) \qquad \cdots \quad \text{(ii)}$$

$$[\text{ where } \lambda = \sqrt{n^2 - \tfrac{1}{4}k^2}\ ]$$

A particular integral of (i)

$$= \frac{1}{D^2 + kD + n^2} F(\cos pt) = F \frac{1}{-p^2 + n^2 + kD} (\cos pt)$$

$$= F\frac{(n^2-p^2)-kD}{(n^2-p^2)^2-k^2D^2}(\cos pt)$$

$$= F\frac{(n^2-p^2)\cos pt+kp\sin pt}{(n^2-p^2)^2+k^2p^2}.$$

Hence the most general solution of (i) is

$$x = Ae^{-\frac{1}{2}kt}\cos\left(\sqrt{n^2-\tfrac{1}{4}k^2}\,t+\epsilon\right) + F\frac{(n^2+p^2)\cos pt+kp\sin pt}{(n^2-p^2)^2+k^2p^2} \quad \cdots \quad \text{(iii)}$$

where A and ϵ are integration constants whose values are to be determined from given initial conditions.

Putting $\tan\epsilon' = \dfrac{kp}{n^2-p^2}$, the second term on the right hand side of (iii) reduced to

$$F\frac{kp\cot\epsilon'\cos pt+kp\sin pt}{k^2p^2\cot^2\epsilon'+k^2p^2} = \frac{F\sin\epsilon'}{kp}\cos(pt-\epsilon').$$

Thus, (iii) gives,

$$x = Ae^{-\frac{1}{2}kt}\cos\left(\sqrt{n^2-\tfrac{1}{4}k^2}\,t+\epsilon\right) + \frac{F\sin\epsilon'}{kp}\cos(pt-\epsilon'). \quad \cdots \quad \text{(iv)}$$

This shows that the motion is a composition of the two simple harmonic motions of periods $\dfrac{2\pi}{\sqrt{n^2-\frac{1}{4}k^2}}$ and $\dfrac{2\pi}{p}$ respectively, the amplitude of the first, namely $Ae^{-\frac{1}{2}kt}$ continually diminishing with t. The free harmonic motion therefore ultimately dies out, and the particle ultimately performs a forced oscillation of the same period as of the periodic disturbing force.

In the particular case where $p = n$, $\epsilon' = \dfrac{\pi}{2}$, and the part of the solution due to forced oscillation is $x = \dfrac{F}{kn}\sin nt$.

If k is small, this leads to an oscillation with a large amplitude.

Miscellaneous Examples

1. An aeroplane has a speed of v miles per hour and carries fuel just sufficient for a range of action (out and home) of R miles in calm weather. Prove that in a north wind of ω miles per hour, its range of action is

$$\frac{R(v^2 - \omega^2)}{v(v^2 - \omega^2 \sin^2\phi)^{\frac{1}{2}}}$$

in a direction whose true bearing is ϕ.

2. In order to cross a river in a straight line from a point on one bank to a point on the other bank, a boat takes 10 minutes and to do the return journey, it takes 20 minutes. The current flows at 3 miles per hour and the speed of the boat relative to the water is 6 miles per hour. Show that the breadth of the river is $\frac{1}{4}\sqrt{(15)}$ miles.

3. If the co-ordinates of a moving point at time t are given by $x = At^2 + B$, $y = Ct$, show that the path is a parabola.

4. (i) A particle is moving with constant velocity parallel to the x-axis and a velocity proportional to x parallel to the y-axis. Show that the path is a parabola.

(ii) If the position of a moving point at time t be $x = a \cos kt$, $y = a \sin kt$, show that the point moves in a circle.

5. The position of a moving point P at time t is given by $x = a(2t + \sin 2t)$, $y = a(1 - \cos 2t)$; show that the acceleration of P is constant and equal to $4a$ and its direction makes an angle t with the x-axis.

6. If the velocity of a point moving in a plane curve varies as the radius of a curvature, show that the direction of motion revolves with constant angular velocity.

7. A bullet fired from a gun and moving horizontally passes through 3 screens hanging vertically at equal distances of x foot and the times of passing through the screens are t_1, t_2, t_3 measured from the moment the bullet was fired. If f be the uniform retardation, show that

$$f = \frac{2x(t_1 - 2t_2 + t_3)}{(t_2 - t_3)(t_3 - t_1)(t_1 - t_2)}.$$

8. A particle is projected from the point O with velocity u at an angle α $\left(\alpha \neq \frac{\pi}{2}\right)$ with the horizon. Two perpendicular lines OX, OY through O are taken as axes of reference.

(*a*) Show that there are in general two different directions in which a particle can be projected from O to strike a point $P(h, k)$.

(*b*) Show that the least velocity of projection for a projectile, to hit a given point (h, k) is

$$[g\{k + \sqrt{(h^2 + k^2)}\}]^{\frac{1}{2}}.$$

(*c*) If t_1, t_2 be two different times in going from O to P by different paths, show that the product of two times is independent of the velocity of projection.

9. If t_1 and t_2 be the times of flight from a point O to a point P and if OP be inclined to the horizontal line OX at an angle β, show that

$$t_1^2 + 2t_1t_2 \sin\beta + t_2^2$$

is independent of β.

10. A particle is projected with velocity $2\sqrt{(hg)}$ so that it just clears two posts of equal height h, which are at a distance of $2h$ from each other. Show that the latus rectum of the parabolic path is equal to the distance between the two posts.

11. A particle projected from a point with velocity u, has a horizontal range H at times t_1, t_2. Show that t_1^2, t_2^2 satisfy the equation

$$g^2t^4 - 4u^2t^2 + 4H^2 = 0.$$

12. A shell bursts on contact with the ground and pieces from it fly in all directions with velocities up to 80 feet per second. Show that a man 100 ft. away is in danger for $\frac{5}{2}\sqrt{2}$ seconds.

[*Compare Ex. 8 of Art. 8·10. Use here Art. 8·2, Cor. 2*]

13. A shot is fired with velocity u from the top of a tower h ft. high and strikes the ground at a distance k ft. from the foot of the tower. Show that the possible times of flight are the roots of the equation

$$g^2t^4 - 4(gh + u^2)\,t^2 + 4(h^2 + k^2) = 0.$$

14. If at any point of a parabolic path, the velocity be v and its inclination to the horizon be β, show that the particle is moving perpendicular to its former direction after a time $v/g \sin\beta$.

15. A particle is projected from the point O with velocity u at an angle a to the horizontal direction. Show that it will be at the ends of the latus rectum of the parabolic path at times

$$\frac{u}{g}(\sin a \pm \cos a).$$

16. Two straight lines inclined at an angle a meet at O. Two masses move at a uniform rate along them starting from O. Show that their centre of inertia describes a straight line with uniform velocity.

17. Two straight lines OA and OB inclined at an angle θ meet at O. If two masses mk and m (k being constant) start simultaneously from O, with velocities u and ku, show that the internal bisector of the angle AOB is the locus of their centre of inertia.

18. Two masses m_1, and m_2 $(m_1 > m_2)$ are connected by a light inextensible string passing over a light smooth pulley and are allowed to hang freely. Show that the acceleration of the centre of inertia downwards is

$$\left(\frac{m_1 - m_2}{m_1 + m_2}\right)^2 g.$$

19. PAQ is the arc cut off by a focal chord PSQ from the parabolic path described by a projectile. If T be the time of describing the arc PAQ, show that it is equal to the time of falling from rest through a height equal to the length of the chord.

20. Three perfectly elastic balls of masses a, b, c lie on a straight line on a plane and a is projected towards b. If the velocity of a after striking b is equal to that of b after striking c, prove that

$$(a+b)(b+c) = 4ac.$$

21. A particle is projected from a point A with a velocity V at an angle of elevation a so as to pass through another point B whose co-ordinates referred to A as origin are (x, y) the axis of y being vertical. Prove that $\tan a$ is given by the quadratic equation

$$\tan^2 a - \frac{2V^2}{gx} \tan a + 1 + \frac{2V^2 y}{gx^2} = 0$$

and the two times of transits are the positive roots of

$$g^2 t^4 - 4(V^2 - gy)\, t^2 + 4r^2 = 0,$$

where r is the distance of B from A and g is the acceleration due to gravity. Prove also that the product of the times of transit is independent of V and is equal to the square of the time occupied by a particle in falling from rest vertically through a distance equal to AB.

22. A particle is moving in a straight line with an acceleration μx towards a fixed centre in the straight line and is also subjected to an additional acceleratian $L \cos pt$.

Find the motion assuming that the particle starts from rest at a distance b from the fixed centre.

23. A body of mass m, moves on a horizontal table being attached to a fixed point on the table by an extensible string whose modulus of elasticity is λ; if the unstretched length of the string be a, find the speed of the particle when it is describing a circle of radius r.

24. An artificial satellite is girdling the earth in a circular orbit of radius R at a height where the gravitational acceleration is g'. Show that its period (neglecting all effects of earth's rotation) is

$$2\pi (R/g')^{\frac{1}{2}}.$$

25. A ball is thrown from the ground level to strike a vertical wall at a distance x away at a height y above the ground, when travelling horizontally. Show that the velocity of projection is

$$\left\{g\left(2y+\frac{x^2}{2y}\right)\right\}^{\frac{1}{2}}.$$

26. A car is travelling steadily at 60 m.p.h. round a banked track in a circle of radius of 220 yards. Find the acceleration of the car and show that there will be no tendency to slip if the track is banked at an angle of $\tan^{-1}\frac{11}{30}$.

27. A ship steams due West at the rate of 15 Kms. per hour when the river is flowing at the rate of 6 Kms. per hour due South. What is the velocity relative to the ship of a train going due North at the rate of 30 Kms. per hour?

28. The components of the velocity of a particle of mass m parallel to the rectangular axes OX, OY are $(hx+by)$ and $-(ax+hy)$ respectively where a, b, h are constants. Find the force-components and also the path of the particle.

29. If two smooth uniform spherical balls of masses m and m' moving with velocities u and u' respectively, impinge directly, prove that the condition that each loses the same amount of kinetic energy is

$$(3+e)(mu+m'u')+(1-e)(mu'+m'u)=0$$

where e is coefficient of restitution.

30. Two balls of elasticity e moving in opposite directions collide directly. If the mass of the first ball be twice that of the second and the velocity of the second ball be twice that of the first and if E_1 and E_2 denote the sums of the kinetic energies of the two balls before and after impact respectively, show that

$$e=\sqrt{\left(\frac{E_2}{E_1}\right)}.$$

31. A particle moving with uniform acceleration in a straight line passes points P, Q, R. If $PQ=QR=b$ and the time from P to Q is a and that from Q to R is c, prove that the acceleration is $\dfrac{2b(a-c)}{ac(a+c)}$.

32. A particle is thrown from a point distant a from a smooth vertical wall against the wall and returns to the point of projection. If e be the coefficient of elasticity, u the velocity of projection, $45°$, the elevation of projection and t be the total time flight, then

$$u=\{ag(e+1)/e\}^{\frac{1}{2}},\ t=\{2a(e+1)/ge\}^{\frac{1}{2}}.$$

33. A particle projected vertically upwards attains heights x_1, x_2, x_3 above the initial position with velocities v_1, v_2, v_3 respectively; show that the acceleration due to gravity is

$$\frac{v_1^2x_1(x_2-x_3)+v_2^2x_2(x_3-x_1)+v_3^2x_3(x_1-x_2)}{2(x_2-x_3)(x_3-x_1)(x_2-x_1)}.$$

34. Three balls A, B, C of masses m_1, m_2, m_3 respectively lie on a straight line. A projected towards B with velocity u impinges directly on B and B afterwards impinges on C. Show that the velocity of C after impact is

$$\frac{m_1 m_2 (1+e)^2}{(m_1+m_2)(m_2+m_3)} u,$$

e being the coefficient of elasticity in either case.

35. A smooth billiard ball impinges on another equal ball of same mass and radius in a direction that makes an angle α with the line of centres at the moment of impact and e is their coefficient of restitution. Prove that the angle through which the direction of motion of the impinging ball is deviated, is

$$\tan^{-1}\left[\frac{(1+e)\tan\alpha}{1-e+2\tan^2\alpha}\right],$$

36. Over a small smooth pulley is placed a uniform flexible cord ; the system is initially at rest and lengths $l-a$ and $l+a$ hang down on the two sides. The pulley is now made to move up with uniform vertical acceleration f. Show that the string will leave the pulley after a time

$$\sqrt{\left(\frac{l}{f+g}\right)} \cosh^{-1}\frac{l}{a}.$$

ANSWERS

22. $x = \left\{b - \frac{L}{\mu - p^2}\right\} \cos\sqrt{\mu}t + \frac{L}{\mu - p^2} \cos pt$. The motion of the particles is thus made of two S.H.M. of period $\frac{2\pi}{\sqrt{\mu}}$ and $\frac{2\pi}{p}$.

23. $\sqrt{\frac{\lambda r}{ma}}(r-a)$. **26.** 11[illegible] ft./sec². **27.** 39 K.M. per hour at an angle $\tan^{-1}$([illegible]) with the North. **28.** $m + Hx$, $m + Hy$ where $H = h^2 - ab$. $ax^2 + 2hxy + by^2 = c$ (a conic section).

UNIVERSITY QUESTIONS

CALCUTTA

1. (a) A particle, moving with uniform acceleration, describes in the last second of its motion $\frac{1}{17}$ths of the whole distance. It starts from rest, find how long it was in motion and through what distance did it move, if it described 6 cms. in the first second.

(b) A ship steams due west at the rate of 15 kms. per hour when the river is flowing at the rate of 6 kms. per hour due south. What is the velocity relative to the ship of a train going due north at the rate of 30 kms. per hour ?

2. (a) A particle of mass m falls from rest at a height h above the ground ; shew that the sum of its potential and kinetic energies is constant throughout the motion.

(b) Two rough planes, inclined at 30° and 60° to the horizon and of the same height, are placed back to back ; masses 5 and 10 gms. are placed on the planes and connected by a string passing over the top of the planes ; if the coefficient of friction be $1/\sqrt{3}$, find the resulting acceleration, assuming that the smaller mass is placed on the plane inclined at 30° to the horizon.

3. (a) Shew that for a projectile the directions of projection, for attaining maximum range on given plane, bisects the angle between the vertical and the inclined plane, the resistance of air being neglected.

(b) Two balls are projected from the same point in directions inclined at 60° and 30° to the horizontal ; if they attain the same height, what is the ratio of their velocities of projection ?

4. (a) A particle is moving in a straight line with an acceleration μx towards a fixed centre in the straight line and is also subjected to an additional acceleration $L \cos pt$; find the motion assuming that the particle starts from rest at a distance b from the fixed centre.

(b) A body of mass m, moves on a horizontal table being attached to a fixed point on the table by an extensible string whose modulus of elasticity is λ ; if the unstretched length of the string be a, find the speed of the particle when it is describing a circle of radius r.

1. (*a*) For a particle moving along a straight line with uniform acceleration find the distance travelled in the n-th second.

(*b*) A particle moving with uniform acceleration in a straight line passes points P, Q, R. If $PQ = QR = b$, and the time from P to Q is a and that from Q to R is c, prove that the acceleration is

$$\frac{2b\,(a-c)}{ac\,(a+c)}.$$

2. (*a*) Two particles of masses m_1 and m_2 ($m_1 > m_2$) are connected by a light inextensible string passing over a light smooth pulley and are allowed to hang freely. Find the acceleration of the system and the tension of the string.

(*b*) A body of mass $(m_1 + m_2)$ is split into two parts of masses m_1 and m_2 by an internal explosion which generates kinetic energy E. Show that, if after explosion the parts move in the same line as before, their relative speed is

$$\sqrt{\left[\frac{2E\,(m_1+m_2)}{m_1 m_2}\right]}$$

3. A particle is projected from a point A with velocity V at an angle of elevation a so as to pass through another point B whose coordinates referred to A as origin are (x, y), the axis of y being vertical. Prove that $\tan a$ is given by the quadratic equation

$$\tan^2 a - \frac{2V^2}{gx}\tan a + 1 + \frac{2V^2 y}{gx^2} = 0$$

and that the two times of transits are the positive roots of

$$g^2 t^4 - 4\,(V^2 - gy)\,t^2 + 4r^2 = 0$$

where r is the distance of B from A, and g is the acceleration due to gravity.

Prove also that the product of the times of transit is independent of V and is equal to the square of the time occupied by a particle in falling from rest vertically through a distance equal to AB.

4. If the displacement of a moving point at any time is given by an equation of the form $x = a\cos kt + b\sin kt$, show that the point executes simple harmonic motion.

If $a = 3$, $b = 4$, $k = 2$, find the period, amplitude, maximum velocity and maximum acceleration of the motion. A particle moving in a

straight line with S.H.M. has velocities v_1 and v_2 when its distances from the centre are x_1 and x_2. Show that the period of motion is

$$2\pi\sqrt{\frac{x_1^2-x_2^2}{v_2^2-v_1^2}}.$$

1. (a) A particle moves along a straight line with uniform acceleration f. If u and v be its initial and final velocities respectively, and s the distance described, prove that $v^2=u^2+2fs$.

(b) A particle starts with initial velocity u and passes successively over the two halves of a given distance with acceleration f_1 and f_2 respectively. Show that the final velocity is the same as if the whole distance were traversed with uniform acceleration $\frac{1}{2}(f_1+f_2)$.

2. (a) Enunciate Newton's Second Law of Motion and deduce the formula $P=mf$.

(b) A railway train whose mass is 100 tons, moving at the rate of sixty miles per hour in a straight line, is brought to rest in 10 seconds by the application of a uniform force. Find how far the train moves during the time for which the force is applied and calculate the magnitude of this force.

3. (a) Prove that the path of a projectile in vacuo is a parabola.

(b) What is an impulsive force? How is it measured?

A bullet weighing half an ounce, leaves the muzzle of a rifle barrel, 2 feet long with a velocity of 2000 ft. per second. Find the force acting on the bullet in the barrel, assuming it to be uniform and the time taken by the bullet to traverse the barrel.

4. (a) A particle describes a circle of a radius r with uniform speed v, find its acceleration.

(b) A train whose weight is 100 tons is moving up an inclined plane with a uniform speed of 45 miles per hour, the inclination being 1 in 100. Find the horse power of the engine, the resistance due to friction etc. being 1/50 of the weight.

1. (a) A particle is moving in a straight line with uniform acceleration f. If its initial velocity is u and it traverses the distance s in time t, prove that $s=ut+\frac{1}{2}ft^2$.

(*c*) A railway train starts from rest at one station and comes to rest at another station. It moves during the first part of the journey with uniform acceleration f; when steam is shut off and the brakes are applied, it moves with uniform retardation f'. If a be the distance between the stations, show that the time the train takes is,

$$\sqrt{2a\frac{f+f'}{ff'}}.$$

2. (*a*) State Newton's Laws of Motion.

(*b*) A jet of water is projected against a wall so that it strikes the wall horizontally with a velocity of 32 ft. per sec. If the diameter of the pipe from which water is issued be $1\frac{3}{4}$ inches, find the pressure on the wall. $(\pi = \frac{22}{7})$

3. (*a*) A particle is projected horizontally with velocity u at an angle a to the horizontal plane. Find the range on a given inclined plane through the point of projection, if the inclination of the plane be β.

(*b*) Define 'Kinetic energy' and 'Potential energy'.

Prove that if a particle moves in a straight line with uniform acceleration, increase of kinetic energy is equal to the work done by the impressed force.

4. If the displacement of a moving point at any time is given by an equation of a form

$$x = a \cos kt + b \sin kt$$

show that the point executes simple harmonic motion.

If $a = 5$, $b = 12$, $k = 4$, find the period, amplitude, maximum velocity and maximum acceleration of the motion.

A particle moving in a straight line with S.H.M. has velocities v_1 and v_2 when its distances from the centre are x_1 and x_2. Show that the period of motion is

$$2\pi\sqrt{\frac{x_1^2 - x_2^2}{v_2^2 - v_1^2}},$$

BURDWAN UNIVERSITY PAPERS

1. (a) For motion in a straight line with uniform acceleration, prove that $v^2 = u^2 + 2fs$.

A particle while moving along a straight line under a uniform acceleration f, passes through the points A_1 and A_2 of its path with velocities u_1 and u_2 respectively, prove that $\dfrac{u_2^2 + u_1^2}{A_1 A_2} = 2f$.

(b) A boat A is sailing with velocity u due North and another B, with a velocity v due West. If initially A were at a distance x West of B, find when they will be nearest to each other and calculate also their shortest distance.

2. (a) Establish the formula $P = mf$ from Newton's Second Law of Motion.

(b) The velocity of a projectile at any two points of its path are v and v'; show that the difference of their altitudes above a horizontal plane is $\dfrac{v^2 - v'^2}{2g}$.

3. (a) A bullet of mass 4 oz. is fired into a target with a velocity of 1200 ft. per second and gets embedded in it. The mass of the target is 20 lbs. and it is free to move. Find the loss of K.E. in ft./lbs.

(b) Two spheres of masses M and m moving with velocities U and u respectively in the same direction impinge directly. If e be the coefficient of restitution, prove that the loss of kinetic energy due to impact is

$$\frac{1-e^2}{2} \frac{Mm}{M+m} (U-u)^2.$$

4. (a) A particle P is moving in a circle of radius a, centre C with uniform speed u. AB is a diameter of the circle and $AP = r$. Find the angular velocity of P about A, B and C.

(b) A particle of mass m is moving along the axis of x under a central force μmx to the origin. Prove that the time to reach the origin is the same whatever be the distance of the starting point from the origin, assuming that it starts from rest.

1. (*a*) A particle is moving in a straight line with uniform acceleration f. Prove that the space described in t seconds is $ut+\frac{1}{2}ft^2$, where u is the initial velocity.

Hence or otherwise find the distance traversed in nth second of its motion.

(*b*) AB is the horizontal diameter of a vertical circle. Show that the time taken by a particle to slide down any chord AC varies inversely as the time down the chord BC.

2. (*a*) Two particles of masses m_1 and m_2 are connected by a light inextensible string which passes over a fixed light smooth pulley. Find the acceleration of the system and the tension of the string.

(*b*) A particle is projected from a point with a velocity u in a direction, making angle α with the horizon. Find its position after t seconds. Find also the horizontal range of the particle and the time of flight.

3. (*a*) A particle is moving along a straight line under the action of a constant force acting along the line. Prove that the increase in the kinetic energy of the particle in any time is equal to the work done by the external force during that time.

(*b*) A sphere impinges obliquely on another sphere at rest. If the spheres are smooth, perfectly elastic and equal in mass, prove that they will move at right angles after impact.

4. (*a*) A particle moves in a circle of radius a with uniform speed v. Show that the acceleration of the particle is $\frac{v^2}{a}$ towards the centre.

(*b*) A particle of mass m moves along the axis of x under a central force $m\mu x$ towards the origin. When $t=2$ seconds, it passes through the origin, and when $t=4$ seconds, its velocity is $4 ft/sec$. If the complete period is 16 seconds, prove that the amplitude of the path is $\frac{32\sqrt{2}}{\pi}$ ft.

NORTH BENGAL UNIVERSITY PAPERS

1. To a man walking at the rate of 5 kilometers per hour, rain appears to fall vertically; find the actual direction and speed of the rain, given that its apparent speed is $5\sqrt{3}$ kilometers per hour.

If a, b, c be the distances described by a particle during the p^{th}, q^{th}, r^{th} seconds respectively, the particle moving on a straight line with a constant acceleration, prove that

$$a(q-r)+b(r-p)+c(p-q)=0.$$

2. Two particles of masses m_1 and m_2 are connected by a light inelastic string; m_2 is placed on a smooth plane inclined at an angle α to the horizon and m_1 is suspended vertically, the string passing over a small smooth pulley at the top of the plane. If m_1 descends, find the tension of the string.

A projectile returns to the horizontal plane through the point of projection in 4 seconds at a distance of 192 feet from the point of projection. Find the velocity of projection.

3. A particle is moving in a straight line with an acceleration always directed towards and proportional to its distance from a fixed point O on the line and with an additional acceleration $A \cos pt$. Investigate the motion.

1. A particle starts from a point A to move along a straight line with constant acceleration f. If its initial velocity be u, find its distance from A at time t.

The speed of a train increases at a constant rate α from 0 to v, then remains constant for an interval, and finally decreases to 0 at a constant rate β. If l be the total distance described, prove that the total time occupied is

$$\frac{l}{v}+\frac{v}{2}\left(\frac{1}{\alpha}+\frac{1}{\beta}\right).$$

2. Prove that the path of a projectile in vacuum is a parabola.

A gun of mass M fires a shell of mass m horizontally, and the energy

of explosion is such as would be sufficient to project the shell vertically to a height h. Show that the velocity of recoil of the gun is

$$\{2m^2gh/M(M+m)\}^{\frac{1}{2}}.$$

3. A particle is describing a circle of radius a with constant speed Find its acceleration.

If the velocity of a body of mass 100 gms. moving in a straight line increases from 10 (meters)/(sec.) to 50 (meters)/(sec), while moving a distance of 100 meters with uniform acceleration, find the force acting on the body.

GAUHATI UNIVERSITY PAPERS

1. (a) State Newton's Laws of Motion. Establish the energy equation for a particle moving in a straight line with uniform acceleration.

(b) Three screens are placed a feet apart, and a bullet fired through them at t_1, t_2, t_3 recorded by an electric chronograph. If the retardation is assumed uniform, show that its magnitude is

$$\frac{2a(t_3-2t_2+t_1)}{(t_3-t_1)(t_3-t_2)(t_2-t_1)}.$$

2. (a) A particle is projected with velocity u at an angle α with the horizon. Calculate the range on the horizontal plane through the point of projection, the time of flight and the maximum height attained by the particle.

(b) A body projected with the same velocity at two different angles covers the same horizontal range R. If t, t' be two times of flight, prove that

$$R=\frac{g}{2}tt'.$$

3. (a) The speed v of a particle moving along the axis Ox is given by the relation $v^2=n^2(8ax-x^2-12a^2)$.

Prove that the motion is simple harmonic, with amplitude $2a$, and that the time taken from $x=4a$ to $x=6a$ is $\frac{\pi}{2n}$. What is the periodic time?

(b) A particle is performing a simple harmonic motion of period T about a centre O and it passes through a point P with velocity v in the direction OP. Prove that the time which lapses before its return to P is

$$\frac{T}{\pi}\tan^{-1}\frac{vT}{2\pi . OP}.$$

4. (a) Show that if two equal perfectly elastic smooth spheres impinge directly, then they interchange their velocities.

(b) A mass of m lb. falls from rest through h ft. and then brought to rest by penetrating s ft. into sand. Find the average resistance of the sand.

1. (a) The position of a point P moving on a straight line is given by $s = t^3 - 9t + 20$, where t is the time. Discuss the motion of P.

(b) A force equal to the weight of one kilogram acts on a body continuously for 5 seconds and causes it to describe one metre in that time. Find the mass of the body.

2. (a) Define simple harmonic motion.

If v_1, v_2 are the velocities of a particle moving in S.H.M. at distance x_1, x_2 from the centre, show that the time of a complete oscillation is

$$2\pi\sqrt{\left(\frac{x_1^2 - x_2^2}{v_2^2 - v_1^2}\right)}.$$

(b) A particle projected at an angle α with the horizontal from the foot of a plane, whose inclination to the horizon is β strikes the plane at right angles. Show that

$$\cot\beta = 2\tan(\alpha - \beta).$$

3. (a) A gun of mass M fires a shell of mass m horizontally and the energy of explosion is such as would be sufficient to project the shell vertically to a height h. Prove that the velocity of the recoil is

$$m\sqrt{\frac{2gh}{M(M+m)}}.$$

(b) [illegible] perfectly elastic spheres impinge at right angles. [illegible] after impact will still be at right angles.